

# On Eigenvalues of the Sum of Two Random Projections

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**Abstract** We study the behavior of eigenvalues of matrix  $P_N + Q_N$  where  $P_N$  and  $Q_N$  are two  $N$ -by- $N$  random orthogonal projections. We relate the joint eigenvalue distribution of this matrix to the Jacobi matrix ensemble and establish the universal behavior of eigenvalues for large  $N$ . The limiting local behavior of eigenvalues is governed by the sine kernel in the bulk and by either the Bessel or the Airy kernel at the edge depending on parameters. We also study an exceptional case when the local behavior of eigenvalues of  $P_N + Q_N$  is not universal in the usual sense.

**Keywords** Random matrices · Random projections · Eigenvalues · Universality · Tracy–Widom distribution · Sine kernel · Airy kernel

## 1 Introduction and Notation

Let  $A$  and  $B$  be two complex Hermitian matrices, perhaps random, and consider matrix  $X = A + UBU^*$  where  $U$  is a uniformly-distributed unitary matrix. What can be said about the distribution of eigenvalues of  $X$  if the size of the matrices is large? Especially, what can be said about the properties of this distribution on the local scale?

It is known that if  $B$  is a random matrix with Gaussian entries, then the eigenvalues of  $A + B$  can be described as a determinantal process (Brezin–Hikami [3]). Moreover, if in addition  $A$  is a Wigner matrix (that is, a Hermitian matrix with independent upper-diagonal entries) and if the entries of  $A$  and  $B$  are comparable in size, then the local properties of matrix  $A + UBU^*$  are the same as if this matrix were from GUE, Gaussian Unitary Ensemble (Johansson [13]). On the other hand if  $B$  is a multiple of the identity matrix then the eigenvalues of  $A + UBU^*$  are obviously simply the shifted eigenvalues of the matrix  $A$ .

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The main question at stake here is that of the universality of the eigenvalue distribution of  $A + UBU^*$ . The universality hypothesis claims that for a large class of matrices the local correlations of eigenvalues should approach certain specific functions when the size of matrices grows. The choice of these functions should depend on the overall symmetry properties of matrices such as whether they are complex Hermitian or real symmetric, and on the properties of the global limit density of eigenvalues. Roughly speaking, if the global density is positive and smooth at a point  $x$ , then the limit local behavior at  $x$  should be described by the determinantal point process with the sine kernel. If the global density has a square root singularity at  $x$ , then the local behavior around this point should be described by the determinantal point process with the Airy kernel, and so on.

The universality hypothesis originated in pioneering works of Dyson and Mehta. (See, for example, the discussion in [16].) It was verified in various particular cases in [2, 6, 21, 22]. Very recently, universality was proved for the important cases of Wigner matrices and beta ensembles. See [7, 24] and a review in [8].

Now suppose that  $A$  and  $B$  are different from a multiple of the identity matrix. Is it true that the behavior of eigenvalues of the matrix  $A + UBU^*$  is universal?

The goal of this paper is to investigate this question in the case when  $A$  and  $B$  are random orthogonal projections, which we will denote  $P$  and  $Q$ . That is,  $P$  and  $Q$  are Hermitian (or real symmetric) matrices such that  $P^2 = P$  and  $Q^2 = Q$ . We will assume that the matrices  $P$  and  $Q$  are  $N$ -by- $N$ , and their ranks are  $p$  and  $q$  respectively. We will always consider orthogonal projections in this paper so we omit the adjective orthogonal.

We will call a projection matrix  $P$  a *random Hermitian (or real symmetric) projection* if its entries are complex (or real) random variables and their joint distribution is invariant with respect to unitary (or real orthogonal) conjugations.

In a summary, our main finding is that in this particular example the universality works. In most situations the behavior of eigenvalues of  $P + Q$  for large  $N$  is universal in the sense that it is similar to the behavior of the eigenvalues of classical matrix ensembles. The exceptional case is the local behavior of eigenvalues of  $P + Q$  near  $x = 1$  when the ranks of  $P$  and  $Q$  are both approximately  $N/2$ , and we discuss this case below.

The reason why the eigenvalue distribution of  $P + Q$  is much easier to analyze than that of the general case  $A + B$  is based on two facts. First, there is a “trick” that relates the eigenvalues of  $P + Q$  to eigenvalues of  $PQ$ . Essentially this is due to the fact that an algebra generated by two projections has only one- and two-dimensional irreducible representations. Second, it is already known that the spectrum of  $PQP$  can be related to the Jacobi determinantal ensemble of random points.

Unfortunately, it is difficult to extend this method to more general situations. The first key ingredient can in no way be generalized (the universal  $C^*$ -algebra of two projections is a very tame one, but the universal  $C^*$ -algebra of any other two selfadjoint operators is huge), and unlike for  $PQP$ , very little is known for local statistics involving two more complicated random unitarily invariant objects. The only notable exception is the recent paper by A. Metcalfe [17] where some universality for  $PUBU^*P$  is obtained but this is still far away from the problem of  $A + UBU^*$ .

In the case of projections, these two ingredients can be combined to obtain the following result, which we formulate in a slightly greater generality.

**Theorem 1.1** *Suppose  $P$  is a random Hermitian (or real symmetric) projection of rank  $p$ , and  $Q$  is an independent Hermitian (or real symmetric) random projection of rank  $q$ ,  $p \leq q$ . Assume  $p + q \leq N$  and  $\theta \neq 0$ . Then with probability 1,  $P + \theta Q$  has  $a \equiv q - p$  eigenvalues  $\theta$  and  $b \equiv N - p - q$  eigenvalues 0. The density of the absolutely continuous part of the eigenvalue distribution of  $P + \theta Q$  is induced by a transformation of the following densities:*

$$(i) \quad f^{(2)}(x_1, \dots, x_p) = c_2 \prod_{1 \leq i < j \leq p} |x_i - x_j|^2 \prod_{i=1}^p x_i^a (1 - x_i)^b, \quad x_i \in [0, 1], \quad (1)$$

in the Hermitian case, and

$$(ii) \quad f^{(1)}(x_1, \dots, x_p) = c_1 \prod_{1 \leq i < j \leq p} |x_i - x_j| \prod_{i=1}^q x_i^{a'} (1 - x_i)^{b'}, \quad x_i \in [0, 1], \quad (2)$$

in the real symmetric case, with  $a' = (a - 1)/2$  and  $b' = (b - 1)/2$ .

The transformation is two-fold and given by the formula

$$\lambda_i = \frac{1}{2} (1 + \theta \pm \sqrt{(1 - \theta)^2 + 4\theta t_i}).$$

This theorem allows us to reduce the study of spectral properties of  $P + Q$  to the study of the corresponding Jacobi ensembles. In particular, consider a sequence of matrices  $P_N, Q_N$  with increasing size  $N$ . We use notation  $p_N, q_N$  to denote their ranks.

**Corollary 1.2** *Let  $\mathcal{N}(I)$  denote the number of eigenvalues of the Hermitian matrix  $P_N + Q_N$  in an interval  $I$ . Assume that  $p_N/N \rightarrow p > 0$  and  $q_N/N \rightarrow q \geq p$ , as  $N \rightarrow \infty$ , and suppose that  $p + q \leq 1$ . If  $I \subset (0, 1)$  or  $I \subset (1, 2)$  and  $I$  has positive length, then  $\text{Var}(\mathcal{N}(I)) \sim \pi^{-2} \log N$  for large  $N$ . Moreover, the normalized random variable*

$$\frac{\mathcal{N}(I) - \mathbb{E}\mathcal{N}(I)}{\sqrt{\text{Var}(\mathcal{N}(I))}}$$

converges to a standard Gaussian random variable.

Note: The symbol  $\subset$  denotes proper inclusion. A similar result holds for real symmetric matrices  $P_N + Q_N$  with a different value of the limit variance.

*Proof* Let for definiteness  $J = (\alpha, \beta) \subset (0, 1)$  and  $I = (1 + \alpha^2, 1 + \beta^2) \subset (1, 2)$ . Then  $\mathcal{N}(I) = \mathcal{N}'(J)$ , where  $\mathcal{N}'(J)$  denotes the number of points from ensemble (1) in interval  $J$ . This ensemble is determinantal, hence we can apply the theorem due to Costin, Lebowitz, and Soshnikov ([5] and [23]) and conclude that

$$\frac{\mathcal{N}'(J) - \mathbb{E}\mathcal{N}'(J)}{\sqrt{\text{Var}(\mathcal{N}'(J))}}$$

converges to a standard Gaussian random variable, provided that  $\text{Var}(\mathcal{N}'(J))$  grows to infinity.

Under the assumption that  $p_N/N \rightarrow p > 0$  and  $q_N/N \rightarrow q \geq p$ , as  $N \rightarrow \infty$ , and that  $p + q \leq 1$ , it is known that the limit kernel for the corresponding Jacobi ensemble (1) is the sine kernel. See, for example, [4] or [1]. Hence the calculation of the variance can be done similar to the calculation of variance in the case of the Gaussian Unitary Ensemble (see calculations in [11] and related formula (16.1.3) and Appendix A.38 in [16]). This calculation results in the variance asymptotically equal to  $\pi^{-2} \log N$  for large  $N$ .  $\square$

Similarly, the other properties of eigenvalues of the Hermitian matrix  $P_N + Q_N$  in the interval  $J$  are analogous to the properties of the points from the Jacobi ensemble for the simple reason that the transformation  $\lambda \rightarrow (\lambda - 1)^2$  is locally linear on this interval. In particular, in the limit these properties can be described by the sine kernel.

For the behavior at the edge of the support, the results are also similar to the case of the classical ensembles and this behavior can be described by either the Bessel kernel (if

$p + q = 1$ ) or the Airy kernel (if  $p + q < 1$ ). The theorem below gives details for the case  $p + q < 1$ .

**Theorem 1.3** *Let  $P_N$  and  $Q_N$  be two random projections in  $\mathbb{C}^N$  with ranks  $p_N$  and  $q_N$  respectively. Assume that  $\lim_{N \rightarrow \infty} \frac{p_N}{N} = p$  and  $\lim_{N \rightarrow \infty} \frac{q_N}{N} = q$ , and suppose that  $p + q < 1$ . Let  $\lambda_{(1)}$  denote the largest eigenvalue of  $P_N + Q_N$  which is not equal to 2 and let*

$$\mu = \sqrt{q(1 - p)} + \sqrt{p(1 - q)}.$$

Then there exists  $\sigma > 0$  such that for every  $u$

$$\mathbb{P} \left\{ \frac{\lambda_{(1)} - \mu}{\sigma N^{-2/3}} \leq u \right\} \rightarrow F_2(u),$$

where  $F_2(u)$  is the Tracy–Widom distribution function for the edge of the Gaussian unitary ensemble.

This theorem directly follows from recent results about the distribution of the largest eigenvalue in the Jacobi ensemble with changing parameters. A similar result holds for real symmetric projections except that  $\sigma$  is different and the probability converges to  $F_1(u)$  instead of  $F_2(u)$ .

The only unusual situation occurs at  $x = 1$  when this point belongs to the support of the absolutely continuous part of the limit eigenvalue distribution, which occurs when  $p = q = 1/2$ . We will analyze this special situation in the following theorem.

Let  $J_a(z)$  denote the Bessel function with index  $a$ .

**Theorem 1.4** *Let  $P_N$  and  $Q_N$  be two random projections in  $\mathbb{C}^N$  with ranks  $p_N$  and  $q_N$  respectively. Assume that  $q_N - p_N = a \geq 0$ , and  $N - p_N - q_N = b \geq 0$ . Let  $s_N = t/(\sqrt{2}p_N)$ , where  $t > 0$ . Then the expected number of eigenvalues of  $P + Q$  in the interval  $I_N = [1, 1 + s_N]$  is*

$$\mathbb{E}\mathcal{N}(I_N) = \int_0^t x (J_a(x)^2 - J_{a+1}(x)J_{a-1}(x)) dx + o(1)$$

for large  $N$ .

By using the power series for the Bessel functions,

$$J_a(z) = \sum_{m=0}^{\infty} \frac{(-1)^m (z/2)^{2m+a}}{m!(m+a)!},$$

we obtain the following corollary.

**Corollary 1.5** *For large  $N$ ,*

$$\mathbb{E}\mathcal{N}(I_N) \sim \frac{2}{(a+1)!^2} \left(\frac{t}{2}\right)^{2a+2} + R(t),$$

where  $R(t)$  is a differentiable function such that  $\lim_{t \rightarrow 0} R(t)/t^{2a+4}$  exists and finite.

(Here the asymptotic equivalence  $\sim$  is understood relative to increase in  $N$ .) For example, if  $a = 0$ , then  $\mathbb{E}\mathcal{N}(I_N) \sim t^2/2 + R(t)$  where  $\lim_{t \rightarrow 0} R(t)/t^4 < \infty$ .

In order to appreciate the significance of this corollary, note that under assumptions of Theorem 1.4, the eigenvalue distribution of  $P_N + Q_N$  weakly converges (in probability) to a fixed probability distribution which is absolutely continuous on  $(0, 2)$  and has the density

$$f(\lambda) = \frac{1}{\pi} \frac{1}{\sqrt{\lambda(2-\lambda)}}.$$

This can be shown either by using a relation of this ensemble with free probability theory (see [20]) or by computing the limit distribution for the corresponding Jacobi ensemble.

In particular, the limiting density is not zero at  $\lambda = 0$  and does not depend on  $a$  or  $b$ . Hence we can see that the expected number of eigenvalues in the interval  $I_N$  cannot be estimated as  $N \times \text{length of } I_N \times \text{density} + \text{terms of smaller order}$ , which would result in a term of the order  $ct$  with  $c > 0$ . For the ensemble  $P_N + Q_N$ , the expected number of eigenvalues in  $I_N$  is about  $t^2/2$  and significantly smaller than this estimate.

Thus, the local behavior near  $x = 1$  differs strongly from the local behavior of the GUE eigenvalues. In the case of GUE, a similarly scaled limit gives  $\mathbb{E}\mathcal{N}(I_N) \sim \rho(x)t$  where  $\rho(x)$  is the density of the semicircle law. Hence, the result in Corollary 1.5 is in contradiction with the heuristic that the local behavior of eigenvalues of a Hermitian ensemble is similar to that of GUE eigenvalues provided that the limit eigenvalue density is non-zero and smooth at a given point. (For example, at the beginning of Sect. 6.3.3 on p. 113 of [15], it is stated that “In the context of Hermitian matrix models, the universality may be stated as the fact that the global eigenvalue regime determines the local eigenvalue regime.”)

A possible intuitive explanation of this phenomenon is the additional symmetry of the model. This symmetry forces eigenvalues either land exactly on  $x = 1$ , or come in pairs located symmetrically around  $x = 1$ . In both cases the repulsion between eigenvalues ensures that for small  $t$  the interval  $(0, t/2p_N)$  has fewer eigenvalues than the density function predicts.

Another interesting feature of the result in Theorem 1.4 is that the limit does not depend on  $b$ , that is, on the excess of the dimension  $N$  of the ambient space over the total rank of  $P$  and  $Q$ . The only provision is that  $b$  stays constant as  $N$  grows.

Hence the local behavior of eigenvalues near  $x = 1$  exhibits universality with respect to parameter  $b$  but not with respect to parameter  $a$ . The difference between ranks of  $P$  and  $Q$  does matter for the limiting local behavior of eigenvalues near  $x = 1$  despite the fact that it does not influence the global limiting distribution.

For the case of real symmetric matrices we have an analogous result.

**Theorem 1.6** *Let  $P_N$  and  $Q_N$  be two random projections in  $\mathbb{R}^N$  with ranks  $p_N$  and  $q_N$  respectively. Assume that  $q_N - p_N = a \geq 0$ , and  $N - p_N - q_N = b \geq 0$ . Let  $s_N = t/(\sqrt{2}p_N)$ , where  $t > 0$ . Then for large  $N$  the expected number of eigenvalues of  $P + Q$  in the interval  $I_N = [1, 1 + s_N]$  is*

$$\begin{aligned} \mathbb{E}\mathcal{N}(I_N) &= \int_0^t x(J_{a'+1}(x)^2 - J_{a'+2}(x)J_{a'}(x)) dx \\ &\quad + \int_0^t J_{a'+1}(x) \left( \int_0^x J_{a'+1}(u) du - 1 \right) dx + o(1), \end{aligned}$$

where  $a' = (a - 1)/2$ .

By using the power series expansion for the Bessel functions we obtain the following consequence.

**Corollary 1.7** For large  $N$ ,

$$\mathbb{E}\mathcal{N}(I_N) \sim \frac{1}{(a + 3)2^{\frac{a-1}{2}}\Gamma(\frac{a+1}{2})}t^{\frac{a+3}{2}} + R(t),$$

where  $R(t)$  is a differentiable function such that  $\lim_{t \rightarrow 0} R(t)/t^{(a+7)/2}$  exists and finite.

For example for  $a = 0$ ,  $\mathbb{E}\mathcal{N}(I_N) \sim 1/(3\sqrt{\pi/2})t^{3/2} + R(t)$ , where  $\lim_{t \rightarrow 0} R(t)/t^{7/2} < \infty$ .

The rest of the paper organized as follows: we prove Theorem 1.1 in Sect. 3 after proving some results about the relation of eigenvalues of  $P + Q$  and of  $PQP$  in Sect. 2. Section 4 is about the (universal) edge behavior of eigenvalues. Section 5 proves Theorems 1.4 and 1.6 about the exceptional case when the behavior of eigenvalues is not universal. And Sect. 6 concludes.

## 2 Relation of Eigenvalues of $P + Q$ and $PQP$

**Lemma 2.1** Let  $P$  and  $Q$  be orthogonal projections and assume that  $\text{Range } P \cap \text{Range } Q = 0$ . Let  $t_i > 0$  be an eigenvalue of operator  $PQP$ . Then,

$$\lambda_i = \frac{1}{2}(1 + \theta \pm \sqrt{(1 - \theta)^2 + 4\theta t_i})$$

are eigenvalues of  $P + \theta Q$ .

*Proof* Let  $u_i$  be an eigenvector of  $PQP$  with eigenvalue  $t_i$  and define  $v_i := QPu_i$ . Then, we have

$$QPu_i = v_i, \quad \text{and} \quad Pv_i = t_i u_i.$$

Clearly,  $u_i \in \text{Range } P$ ,  $v_i \in \text{Range } Q$ . Hence,  $Pu_i = u_i$  and  $Qv_i = v_i$ .

Define

$$\alpha_i = \frac{1}{2t_i}(-1 + \theta \pm \sqrt{(1 - \theta)^2 + 4\theta t_i}).$$

Then,

$$\begin{aligned} (P + \theta Q)(u_i + \alpha_i v_i) &= u_i + \alpha_i t_i u_i + \theta v_i + \theta \alpha_i v_i = (1 + \alpha_i t_i) \left( u_i + \frac{\theta(1 + \alpha_i)}{\alpha_i(1 + \alpha_i t_i)} \alpha_i v_i \right) \\ &= \lambda_i (u_i + \alpha_i v_i). \end{aligned}$$

Note that  $u_i + \alpha_i v_i \neq 0$  by the assumption that  $\text{Range } P \cap \text{Range } Q = 0$ . Hence  $\lambda_i$  is an eigenvalue of  $(P + \theta Q)$ . □

**Lemma 2.2** Suppose  $P$  is a random projection of rank  $p$ , and  $Q$  is a projection of rank  $q$ ,  $p \leq q$ . Assume  $p + q \leq N$  and  $\theta \neq 0$ . Then, with probability 1, the matrix  $P + \theta Q$  has  $2p$  distinct eigenvalues

$$\lambda_i^\pm = \frac{1}{2}(1 + \theta \pm \sqrt{(1 - \theta)^2 + 4\theta t_i})$$

where  $t_1, \dots, t_p$  are non-zero eigenvalues of operator  $PQP$ ,  $t_i \in (0, 1)$ . In addition, with probability 1,  $P$  has  $q - p$  eigenvalues  $\theta$  and  $N - p - q$  eigenvalues 0.

*Proof* Let  $[L]$  denote the dimension of linear space  $L$ . Then, by using the fact that with probability 1, the range and kernel spaces of  $P$  and  $Q$  are in general position relative to each other, we can calculate:

$$[\text{Ker } P \cap \text{Ker } Q] = N - p - q, \tag{3}$$

$$[\text{Range } P \cap \text{Range } Q] = (p + q - N)_+ = 0, \tag{4}$$

$$[\text{Ker } P \cap \text{Range } Q] = (N - p + q - N)_+ = q - p, \tag{5}$$

$$[\text{Range } P \cap \text{Ker } Q] = (p + N - q - N)_+ = 0, \tag{6}$$

$$[\text{Ker } QP] = [\text{Range } P \cap \text{Ker } Q] + [\text{Ker } P] \tag{7}$$

$$= N - p, \tag{8}$$

$$[\text{Range } QP] = N - [\text{Ker } QP] = p, \tag{9}$$

which implies that  $QP$  has  $p$  positive singular values, and therefore  $PQP$  has  $p$  positive eigenvalues. In addition, (4) implies that all singular values of  $QP$  are smaller than 1.

Therefore, by previous lemma  $Q + P$  has at least  $2p$  eigenvalues (possibly counting with multiplicities) that are distinct from 0 and  $\theta$ .

With probability 1 the singular values of  $QP$  in the interval  $(0, 1)$  are distinct. Indeed, the set  $\mathcal{X}$  of  $P$  such that  $PQP$  has a multiple eigenvalue  $t$  with  $t \in (0, 1)$  is characterized by an algebraic condition and therefore it is algebraically closed. The set of all  $P$  is algebraically irreducible. Hence  $\mathcal{X}$  is either coincide with the set of all  $P$  or has a smaller dimension than the set of all  $P$  and therefore has measure zero. It is possible to construct an example such that all eigenvalues in the interval  $(0, 1)$  are distinct. Therefore the measure of  $\mathcal{X}$  is zero.

Therefore, the corresponding eigenvalues of  $Q + P$  are distinct with probability 1.

In addition  $[\text{Ker } P \cap \text{Ker } Q]$  is an eigenspace of  $Q + P$  with dimension  $N - p - q$ , and eigenvalue 0, and  $[\text{Ker } P \cap \text{Range } Q]$  is an eigenspace of  $Q + P$  with dimension  $q - p$  and eigenvalue  $\theta$ . By counting multiplicities we conclude that we found all eigenvalues.  $\square$

### 3 Relation of Eigenvalues of $P + Q$ and the Jacobi Ensemble

By using known results about the eigenvalues of  $PQP$ , we are able to prove Theorem 1.1. (These eigenvalues have been studied in [4] in connection with free probability.)

*Proof of Theorem 1.1* By a change of coordinates we can assume without loss of generality that  $Q$  is a diagonal matrix that has  $Q_{ii} = 1$  for  $i \leq q$  and  $Q_{ii} = 0$  for  $i > q$ .

We will further assume that  $P$  is a projection on the span of  $p$  independent random vectors  $v_1, \dots, v_p$ . The components of a vector  $v_i$  are independent standard Gaussian variables (real in the symmetric case and complex in the Hermitian case). If  $X$  is an  $N$ -by- $p$  matrix with columns  $v_1, \dots, v_p$ , then we can write:

$$X = \begin{bmatrix} A \\ B \end{bmatrix},$$

where  $A$  is a  $q$ -by- $p$  matrix and  $B$  is an  $(N - q)$ -by- $p$  matrix. After orthonormalization, we get

$$Y = \begin{bmatrix} A \\ B \end{bmatrix} (A'A + B'B)^{-1/2}, \tag{10}$$

and the projection on the span of  $v_1, \dots, v_p$  is

$$P = YY'.$$

Formula (10) implies that the non-zero  $q$ -by- $q$  block of  $QPQ$  equals  $A(A'A + B'B)^{-1}A'$ . Hence, the non-zero eigenvalues of  $QPQ$  correspond to non-zero eigenvalues of a  $p$ -by- $p$  matrix  $M = (A'A + B'B)^{-1}A'A$ . It is known that the joint distribution of eigenvalues of  $M$  follows the law of the so-called Jacobi ensemble with parameters  $(p, N - q, q)$ . (See Sects. 3.6 and 7.2.5 of [10] for more information about the Jacobi ensemble.) Namely, the density for the complex case is

$$f^{(2)}(x_1, \dots, x_p) = c_2 \prod_{1 \leq i < j \leq p} |x_i - x_j|^2 \prod_{i=1}^p x_i^{a_2} (1 - x_i)^{b_2}, \quad x_i \in [0, 1],$$

where  $a_2 = q - p$  and  $b_2 = N - p - q$  (see Sect. 8 in [12]).

The density for the real case is

$$f^{(1)}(x_1, \dots, x_p) = c_1 \prod_{1 \leq i < j \leq p} |x_i - x_j| \prod_{i=1}^p x_i^{a_1} (1 - x_i)^{b_1}, \quad x_i \in [0, 1],$$

where  $a_1 = (q - p - 1)/2$  and  $b_1 = (N - q - p - 1)/2$  (see Muirhead [18, Thm. 3.3.4]).

This completes the proof. □

### 4 Eigenvalues at the Edge

In the first step of this section we establish the limit of the eigenvalue distribution on the global scale. In particular we identify the support of the limit distribution. Then we study the behavior of the eigenvalues at the edge by using the connection between eigenvalues of  $P_N + Q_N$  and the Jacobi ensemble. We rely on known results about the edge behavior of eigenvalues from the Jacobi ensemble. As an aside, we derive the limit global density for the Jacobi ensemble with variable parameters.

We abuse notation a bit and denote  $\lim_{N \rightarrow \infty} \frac{pN}{N}$  by  $p$  and  $\lim_{N \rightarrow \infty} \frac{qN}{N}$  by  $q$ . That is, in this section  $p$  and  $q$  denote the limits of fractional ranks of matrices  $P_N$  and  $Q_N$ . We also use the notation

$$\begin{aligned} a &:= q - p \equiv \lim_{N \rightarrow \infty} \frac{a_N}{N}, \\ b &:= 1 - p - q \equiv \lim_{N \rightarrow \infty} \frac{b_N}{N}, \quad \text{and} \\ c &:= p(1 - q) + q(1 - p) \equiv (1 + a^2 - b^2)/2. \end{aligned}$$

Let  $\lambda_i$  be ordered eigenvalues of  $P_N + Q_N$ , counted with multiplicity. Let

$$\mu_N := \frac{1}{N} \sum_{i=1}^N \delta_{\lambda_i};$$

that is,  $\mu_N$  is the eigenvalue distribution of  $P_N + Q_N$ .

**Proposition 4.1** *With probability 1 the eigenvalue distribution of  $P_N + Q_N$  weakly converges to a probability measure  $\mu$ . The absolutely continuous part of this measure is supported on  $I_1 \cup I_2$ , where  $I_1 = 1 + I$ ,  $I_2 = 1 - I$ , and*

$$I = \left[ \left[ \sqrt{q(1-p)} - \sqrt{p(1-q)} \right], \sqrt{q(1-p)} + \sqrt{p(1-q)} \right].$$



The density of this part is given by

$$\rho(x) = \frac{1}{\pi} \frac{\sqrt{-a^2 + 2c(x-1)^2 - (x-1)^4}}{x(x-1)(x-2)}.$$

In addition,  $\mu$  has an atom with weight  $|a|$  at  $x = 1$ . If  $b > 0$  then  $\mu$  has an atom with weight  $b$  at  $x = 0$ ; and if  $b < 0$  then  $\mu$  has an atom with weight  $-b$  at  $x = 2$ .

*Proof* This result is a particular case of a theorem in [20], with the limit distribution computed by using the standard techniques from free probability. □

As a consequence, we can write down the limit level distribution for the Jacobi ensemble with a changing weight function. (This limit level distribution is known and goes back to the research of Wachter in [25]. However, we obtain it by using free probability calculations and this seems to be new.)

Let

$$f^{(2)}(x_1, \dots, x_N) = c_2 \prod_{1 \leq i < j \leq N} |x_i - x_j|^2 \prod_{i=1}^N x_i^{s_N} (1 - x_i)^{t_N}, \quad x_i \in [0, 1]. \tag{11}$$

Recall that the level density of this distribution is simply the marginal distribution:

$$p_N^{(2)}(x; s_N, t_N) = \int f^{(2)}(x, x_2, \dots, x_N) dx_2 \dots dx_N.$$

**Proposition 4.2** *Let  $p_N^{(2)}(x; s_N, t_N)$  is the level density of the Hermitian Jacobi ensemble with parameters  $s_N$  and  $t_N$ . Assume that  $s_N/N \rightarrow s \geq 0$  and  $t_N/N \rightarrow t \geq 0$  as  $N \rightarrow \infty$ . Define:*

$$\begin{aligned} a &:= \frac{s}{2 + t + s}, \\ c &:= \frac{1}{2} \left( 1 + \frac{s^2 - t^2}{(2 + t + s)^2} \right), \quad \text{and} \\ d &:= \frac{1}{(2 + t + s)^2} \sqrt{(1 + s)(1 + t)(1 + t + s)}. \end{aligned}$$

*Then the level density  $p_N^{(2)}(x; s_N, t_N)$  converges to the limit  $p^{(2)}(x; s, t)$  which is supported on the interval*

$$I = [c - d, c + d].$$

*Its density is*

$$\rho^{(2)}(x) = \frac{2 + t + s}{2\pi} \frac{\sqrt{-a^2 + 2cx - x^2}}{x(x-1)}.$$

*Proof* We use the correspondence between the Jacobi ensemble and the eigenvalues of the sum of random projections  $P_N + Q_N$  established in Theorem 1.1. For this we use projections  $P_N$  and  $Q_N$  with ranks  $p_N$  and  $q_N$  and make sure that  $p_N/N$  and  $q_N/N$  approach  $p$  and  $q$ , which are defined as follows:

$$p := \frac{1}{2 + t + s} \quad \text{and} \quad q := \frac{1 + s}{2 + t + s}.$$

By using Theorem 1.1, it is easy to check that the limit eigenvalue density of this matrix ensemble must coincide with the limit level density of the Jacobi ensemble. Then, the conclusion of the theorem follows from Proposition 4.1.  $\square$

Now let us find out what is the edge behavior of eigenvalues of  $P_N + Q_N$ . The edge behavior of the Jacobi ensemble with variable parameters was analyzed in [4] and [14]. (The case with the fixed parameters was studied in [9] and [19].)

In particular, Johnstone found that the distribution of the largest eigenvalue is concentrated near a point  $x$ , which he defines as follows.

Let

$$\cos \phi = \frac{t - s}{2 + t + s} \quad \text{and} \quad \cos \gamma = \frac{t + s}{2 + t + s}.$$

Then,

$$x = \frac{1 - \cos(\phi + \gamma)}{2}.$$

(We change Johnstone’s notations a bit to adjust them to our situation since we consider the Jacobi ensemble on  $[0, 1]$  instead of  $[-1, 1]$ . In addition, we omitted some terms of order  $O(N^{-1})$  which are important for statistical applications but irrelevant from the asymptotic point of view.)

An easy verification shows that  $x$  coincides with the upper bound of the limiting distribution support which we derived in Proposition 4.2, that is,  $x = c + d$ .

Johnstone proved the following result (see Theorem 3 in [14] and a related result in Theorem 4.15 of [4]):

**Theorem 4.3** (Johnstone) *Assume that  $t > 0$  and let  $x_{(1)}$  denote the largest point from  $N$  points sampled according to distribution (11). Let  $s_N/N \rightarrow s \geq 0$  and  $t_N/N \rightarrow t \geq 0$  as  $N \rightarrow \infty$ . Then, there exists  $\sigma > 0$ , such that for every  $u$ ,*

$$\mathbb{P} \left\{ \frac{\tanh^{-1}(x_{(1)}) - \tanh^{-1}(x)}{\sigma N^{-2/3}} \leq u \right\} \rightarrow F_2(u),$$

where  $F_2(u)$  is the Tracy–Widom distribution function for the edge of the Gaussian unitary ensemble.

Johnstone gives an explicit expression for  $\sigma$  and an explicit bound for the error in this convergence result. He uses a slightly different definition of  $x$  which improves the second order convergence properties. Also note that while the inverse hyperbolic tangent improves the convergence properties for relatively small  $N$ , it is irrelevant from the asymptotic point of view. A similar theorem was proved by Johnstone for the real symmetric Jacobi ensemble, with a different  $\sigma$  and  $F_1(u)$  instead of  $F_2(u)$ .

*Proof of Theorem 1.3* This result follows immediately from Theorems 1.1 and 4.3.  $\square$

### 5 Eigenvalues of $P + Q$ near $x = 1$

#### 5.1 Hermitian Case

The density

$$f(x_1, \dots, x_n) = c \prod_{1 \leq i < j \leq n} |x_i - x_j|^2 \prod_{i=1}^n x_i^a (1 - x_i)^b, \quad x_i \in [0, 1] \tag{12}$$

can be written as a determinant:

$$\begin{aligned} f(x_1, \dots, x_n) &= \frac{1}{N!} \det(K_n(x_i, x_j))_{i,j=1\dots n}; \\ K_n(x, y) &= \sum_{k=0}^{n-1} Q_k(x) Q_k(y) \\ &= c_n \frac{Q_n(x) Q_{n-1}(y) - Q_{n-1}(x) Q_n(y)}{x - y}, \end{aligned}$$

where  $Q_k(x)$  are orthonormal polynomials with respect to the weight  $x^a(1 - x)^b$  on the interval  $[0, 1]$  and  $c_n$  is an appropriate constant.

**Lemma 5.1** *We have*

$$\lim_{n \rightarrow \infty} \frac{1}{2n^2} K_n \left( \frac{t^2}{2n^2}, \frac{t^2}{2n^2} \right) = \frac{1}{2} (J_a(t)^2 - J_{a+1}(t) J_{a-1}(t)),$$

where  $J_a(t)$  denotes the Bessel function of index  $a$ .

*Proof* This is a consequence of a remark in Sect. 7.2.5 and Proposition 7.2.1 in [10]. □

*Proof of Theorem 1.4* By using the relation between the eigenvalues of  $P_N + Q_N$  and of the Jacobi ensemble, we have  $\mathbb{E}\mathcal{N}(I_N) = \mathbb{E}\mathcal{N}'(J_N)$ , where  $\mathbb{E}\mathcal{N}'(J_N)$  is the expected number of eigenvalues of the Jacobi ensemble in the interval  $J_N = [0, t_0^2/(2p_N^2)]$ . Hence, by using Lemma 5.1, we can write:

$$\begin{aligned} \mathbb{E}\mathcal{N}(I_N) &= \int_0^{t_0^2/(2p_N^2)} K_{p_N}(x, x) dx \\ &= \int_0^{t_0} \frac{t}{p_N^2} K_{p_N} \left( \frac{t^2}{2p_N^2}, \frac{t^2}{2p_N^2} \right) dt \\ &= \int_0^{t_0} t (J_a(t)^2 - J_{a+1}(t) J_{a-1}(t)) dt + o(1). \end{aligned} \tag{□}$$

#### 5.2 Real Symmetric Case

In the real symmetric case the density of eigenvalue distribution and all correlation functions (i.e., the marginals of this density) can be written using quaternion determinants:

$$f(x_1, \dots, x_n) = \frac{1}{N!} \mathbb{Q} \det(K_n(x_i, x_j))_{i,j=1\dots n},$$

where  $K_n(x_i, x_j)$  is a quaternion function. The matrix representation of this quaternion is

$$K_n(x, y) = \begin{pmatrix} S_n(x, y) & \tilde{I}_n(x, y) \\ D_n(x, y) & S_n(y, x) \end{pmatrix},$$

where

$$D_n(x, y) = \frac{\partial}{\partial x} S_n(x, y),$$

$$\tilde{I}_n(x, y) = - \int_x^y S_n(x, z) dz + \frac{1}{2} \operatorname{sgn}(y - x),$$

and  $S_n(x, y)$  can be defined in terms of skew-orthogonal or orthogonal polynomials. (See Propositions 6.3.2, 6.3.3, and 6.4.3 in [10].)

The expected number of eigenvalues in an interval  $I$  for the Jacobi ensemble can be computed as

$$\int_I S_{p_N}(x, x) dx.$$

The scaling limit for  $S_n(x, y)$  at the edge of the spectrum is given in Sect. 7.9.1 of [10]. We change the formula a bit since we use interval  $[0, 1]$  instead of  $[-1, 1]$

$$\lim_{n \rightarrow \infty} \frac{1}{2n^2} S_n\left(\frac{x^2}{2n^2}, \frac{y^2}{2n^2}\right) =: S^{\text{hard}}(x^2, y^2) = \frac{x}{y} K^{\text{hard}}(x, y)_{a \rightarrow a+1} + \frac{J_{a+1}(y)}{2y} \left(1 - \int_0^x J_{a+1}(v) dv\right),$$

where

$$K^{\text{hard}}(x^2, y^2) = \frac{x J_{a+1}(x) J_a(y) - y J_{a+1}(y) J_a(x)}{(x^2 - y^2)}.$$

In particular,

$$S^{\text{hard}}(x^2, x^2) = \frac{1}{2} (J_{a+1}(x^{1/2})^2 - J_{a+2}(x) J_a(x)) + \frac{J_{a+1}(x)}{2x} \left(1 - \int_0^x J_{a+1}(v) dv\right).$$

Hence, we can write

$$\begin{aligned} \mathbb{E}\mathcal{N}(I_N) &= \int_0^{t_0^2/(2p_N^2)} S_{p_N}(x, x) dx \\ &= \int_0^{t_0} \frac{t}{p_N^2} S_{p_N}\left(\frac{t^2}{2p_N^2}, \frac{t^2}{2p_N^2}\right) dt \\ &= \int_0^{t_0} t (J_{a+1}(t)^2 - J_{a+2}(t) J_a(t)) dt \\ &\quad + \int_0^{t_0} J_{a+1}(t) \left(1 - \int_0^t J_{a+1}(v) dv\right) dt + o(1). \end{aligned}$$

### 6 Conclusion

We showed that the local behavior of eigenvalues of the sum of two random orthogonal projections satisfies the universality paradigm in general, with a single exception. The exception can perhaps be explained by an additional symmetry present in the model.

The method that we use in the paper is based on the combination of two nice old ideas. One of them is the link between  $P_N + Q_N$  and  $P_N Q_N P_N$ , which is routine in functional/matrix analysis, but is perhaps underused in random matrix theory. The other is the existing results about the distribution of eigenvalues of  $P_N Q_N P_N$ . Unfortunately, both ideas are difficult to extend to the case of a general sum  $A_N + U_N B_N U_N^*$ .

Our results give an example in which the universality of local statistics works. However, the problem of eigenvalue spacings for the general case of  $A_N + U_N B_N U_N^*$  is still wide open.

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