Statistics and Probability Letters 81 (2011) 1094–1097

Contents lists available at ScienceDirect

Statistics and Probability Letters

journal homepage: www.elsevier.com/locate/stapro

Relaxation time is monotone in temperature in the mean-field Ising model

Vladislav Kargin

Department of Mathematics, Stanford University, CA 94305, United States

ARTICLE INFO

Article history:
Received 14 October 2010
Received in revised form 24 February 2011
Accepted 1 March 2011
Available online 8 March 2011

Keywords:
Ising model
Glauber dynamics
Spectral gap
Mean-field model
Monotonicity conjecture

ABSTRACT

In this note we consider the Glauber dynamics for the mean-field Ising model, when all couplings are equal and the external field is uniform. It is proved that the relaxation time of the dynamics is monotonically decreasing in temperature.

Let $G = (V, E)$ be a connected graph with n vertices and let $\mathcal{S}$ be the set of all assignments of numbers $+1$ or -1 to vertices in V. It is convenient to write an element of $\mathcal{S}$ as a vector σ with coordinates $\sigma_x = \pm 1$, where $x \in V$.

The Gibbs measure π of the Ising model is a probability measure on $\mathcal{S}$:

$$\pi(\sigma) = \frac{1}{Z} \exp\left\{\sum_{x,y\in V} J_{xy}\sigma_x\sigma_y + \sum_{x\in V} H_x\sigma_x\right\},$$

where Z is a normalization factor, J_{xy} are real non-negative numbers ("couplings"), and H_x are real numbers ("external field"). It is assumed that $J_{xy} = 0$ if x is not connected to y by an edge of the graph.

We use notation $\langle f\rangle$ to denote the average of function f with respect to the Gibbs measure:

$$\langle f\rangle = E_\pi f := \sum_{\sigma\in\mathcal{S}} f(\sigma)\pi(\sigma).$$

The Glauber dynamics is a reversible Markov chain on $\mathcal{S}$ such that the Gibbs measure π is stationary. Specifically, the transition probabilities are as follows. If assignments σ and σ' differ on more than one vertex, then $P(\sigma \to \sigma') = 0$. If they differ on vertex x, then

$$P(\sigma \to \sigma') = \frac{1}{n}\frac{1}{1 + \exp(-2\sigma'_x(S_x + H_x))}, \tag{1}$$

where

$$S_x = \sum_{y\sim x} J_{xy}\sigma_y.$$

E-mail addresses: kargin@stanford.edu, slavakargin@yahoo.com.

0167-7152/$ – see front matter
doi:10.1016/j.spl.2011.03.001

Finally, if $\sigma' = \sigma$, then

$$P(\sigma \to \sigma') = 1 - \sum_{x \in V} P(\sigma \to \widehat{\sigma}^x),$$

where $\widehat{\sigma}^x$ denote the assignment obtained from σ by changing the assignment at vertex x.

Let $\mathcal{L}^2(\mathcal{S})$ be the linear space of all functions on $\mathcal{S}$ with the scalar product

$$\langle f, g\rangle_\pi := \langle fg\rangle = \sum_{\sigma \in \mathcal{S}} f(\sigma) g(\sigma) \pi(\sigma),$$

where π is the Gibbs measure.

Since P is a reversible chain, there is a basis $\{f_\alpha\}$ such that

$$Pf_\alpha = \lambda_\alpha f_\alpha,$$

where $1 = \lambda_1 > \lambda_2 \geq \cdots \geq \lambda_n$, and

$$\langle f_\alpha, f_\beta\rangle_\pi = \delta_{\alpha\beta}.$$

The speed of convergence to the equilibrium is governed to a large extent by the spectral gap $g := 1 - \lambda_2$. It is conjectured that for every connected graph G and every family of non-negative couplings J_{xy},

$$\frac{\partial \lambda_2}{\partial J_{xy}} \geq 0.$$

See, for example, Question 2 on p. 299 in Levin et al. (2009). This conjecture has been verified analytically only in the case of the n-cycle with arbitrary couplings (Nacu, 2003). In this note we verify this conjecture for the case of the mean-field model, in which all the couplings J_{xy} are the same and the external field is uniform. In this case formula (1) becomes

$$P(\sigma \to \sigma') = \frac{1}{n} \frac{1}{1 + \exp\left(-2\sigma'_x \left(J \sum_{y \sim x} \sigma_y + H\right)\right)}, \tag{2}$$

where x is the only vertex at which σ and σ' are different.

In this case, we prove the following result.

Theorem 1. *Let G be a complete graph on n-vertices, let $J_{xy} = J > 0$ for all $x, y \in V$, and let $H_x = H$ for all $x \in V$. Let λ_2 be the second-largest eigenvalue of the Glauber–Ising model. Then, it is true that λ_2 is increasing in J,*

$$\frac{\partial \lambda_2}{\partial J} \geq 0.$$

The relaxation time of the Glauber dynamics is defined as $t_{rel} = (1 - \lambda_2)^{-1}$, and the temperature T is a parameter proportional to J^{-1}. Hence, Theorem 1 has the following corollary.

Corollary 2. *For the mean-field model, the relaxation time t_{tel} is decreasing in T,*

$$\frac{\partial t_{rel}}{\partial T} \leq 0.$$

The Glauber dynamics on the complete graph was studied as early as in Griffiths et al. (1966), where it was shown that the relaxation time is exponentially growing in the number of vertices provided that the temperature is below a critical threshold. Recently, this model has been investigated in Levin et al. (2010) and in Ding et al. (2009) from the point of view of the theory of finite Markov chains. There it was shown that the Diaconis cutoff phenomenon (Diaconis, 1996) holds in this model when the temperature is above the threshold. In addition, these papers investigated the convergence to equilibrium near the critical temperature and in the slow-convergence regime.

Proof of Theorem 1

Lemma 3. *Let M be the transition matrix of a reversible Markov chain with stationary distribution π. Let M depend on parameter J, and let λ be an eigenvalue of M with eigenvector f such that $\langle f, f\rangle_\pi = 1$. Then,*

$$\frac{\partial \lambda}{\partial J} = \left\langle f, \frac{\partial M}{\partial J} f \right\rangle_\pi .$$

Proof of Lemma 3. We have

$$\lambda = \sum_{i,j} f_i \pi_i M_{ij} f_j.$$

Hence,

$$\begin{aligned}\frac{\partial \lambda}{\partial J} &= \sum_{i,j}\left(\frac{\partial f_i}{\partial J}\pi_i M_{ij} f_j + f_i \frac{\partial \pi_i}{\partial J} M_{ij} f_j + f_i \pi_i \frac{\partial M_{ij}}{\partial J} f_j + f_i \pi_i M_{ij} \frac{\partial f_j}{\partial J}\right)\\ &= \lambda \frac{\partial}{\partial J}\left(\sum_i f_i \pi_i f_i\right) + \sum_{i,j} f_i \pi_i \frac{\partial M_{ij}}{\partial J} f_j\\ &= \left\langle f, \frac{\partial M}{\partial J} f\right\rangle_\pi. \quad \square\end{aligned}$$

Hence, it remains to prove that $\langle f, \frac{\partial P}{\partial J} f\rangle_\pi \geq 0$. Instead of proving this inequality for the original Glauber chain, we will define a reduced chain that has the same second eigenvalue $\lambda_2(J)$ and prove the corresponding inequality for this new chain. (This chain is called the magnetization chain in Levin et al. (2010) and Ding et al. (2009).)

In order to define this new chain, note that every permutation of vertices induces a linear transformation on $\mathcal{L}^2(S)$. Because of the symmetry of the mean-field model, the original Glauber chain has an invariant subspace L that consists of the functions in $\mathcal{L}^2(S)$ that are invariant relative to these transformations. The new transition matrix $\widetilde{P}$ is defined as the restriction of the original matrix P to this invariant subspace. In more detail, let $f \in L$ and let f_k be the value of f on configurations with k spins $+1$ and $n-k$ spins -1. We will write f as a vector $(f_0, f_1, \ldots, f_n)$. This is essentially a choice of a basis in L. Then the transition matrix $\widetilde{P}$ with respect to this basis is tridiagonal with the entries

$$\widetilde{P}_{k,k+1} = \frac{n-k}{n}\frac{1}{1+\mathrm{e}^{(n-2k-1)2J-2H}},$$

where $0 \leq k \leq n-1$,

$$\widetilde{P}_{k,k-1} = \frac{k}{n}\frac{1}{1+\mathrm{e}^{-(n-2k+1)2J+2H}},$$

where $1 \leq k \leq n$, and

$$\widetilde{P}_{kk} = 1 - \widetilde{P}_{k,k-1} - \widetilde{P}_{k,k+1},$$

where $0 \leq k \leq n$ and by convention $\widetilde{P}_{0,-1} = \widetilde{P}_{n,n+1} = 0$.

Definition 4. An eigenvector $f = \{f_k\}_{k=0}^n$ of matrix $\widetilde{P}$ is called *increasing* if $f_{k+1} \geq f_k$ for every k. It is called *strictly increasing* if it is increasing and $f_{k+1} > f_k$ for at least one k.

Lemma 5. *Matrix $\widetilde{P}$ has the same second-largest eigenvalue λ_2 as P, and this eigenvalue has a strictly increasing right eigenvector.*

This fact was shown in Ding et al. (2009), in the statement and proof of Proposition 3.9.

Lemma 6. *The second-largest eigenvalue λ_2 of matrix $\widetilde{P}$ has a unique increasing eigenvector modulo a multiplication by a scalar.*

Proof. Let $f = \{f_k\}_{k=0}^n$ be an increasing eigenvector. We will use the following fact from the proof of Proposition 3.9 in Ding et al. (2009): if $f_{k-1} = f_k$, then $f_{k-1} = f_k = 0$. From this fact it follows that if $f_{k-1} = f_k$ and $f_{k+1} \neq 0$, then

$$(Pf)_k = \widetilde{P}_{k,k-1} f_{k-1} + \widetilde{P}_{kk} f_k + \widetilde{P}_{k,k+1} f_{k+1} = \widetilde{P}_{k,k+1} f_{k+1}.$$

Since $(Pf)_k = \lambda_2 f_k = 0$, hence $\widetilde{P}_{k,k+1} = 0$. This contradicts the definition of $\widetilde{P}_{k,k+1}$. Hence $f_k < f_{k+1}$ for every k.

Now, let f and g be two increasing eigenvectors corresponding to λ_2. By what we just proved, $f_k < f_{k+1}$ and $g_k < g_{k+1}$ for every k. Let

$$r = \min_k \frac{f_{k+1} - f_k}{g_{k+1} - g_k}.$$

Then $h = f - rg$ is either a zero vector or an increasing eigenvector of λ_2 such that $h_k = h_{k+1}$ for some k. The latter is impossible and we showed that modulo a multiplication by a scalar there exists only one increasing eigenvector of λ_2. $\square$

The symmetry of the model implies that $g_k := \{-f_{n-k}\}_{k=0}^{n}$ is another strictly increasing eigenvector of $\widetilde{P}$ with eigenvalue λ_2. By the previous lemma $g_k = f_k$, which means that $f_k = -f_{n-k}$. Since the eigenvector is increasing this implies that $f_k \le 0$ for $k \le n/2$ and $f_k \ge 0$ for $k \ge n/2$.

Let us define the following quantities:

$$s_k = \frac{k(n-2k+1)}{n} \frac{1}{1+\cosh[(n-2k+1)2J-2H]},$$

where $0 \le k \le n$. Note that $s_k \ge 0$ for $k \le (n+1)/2$ and $s_k \le 0$ for $k \ge (n+1)/2$.

The matrix $\partial\widetilde{P}/\partial J$ is tridiagonal with entries

$$\frac{\partial}{\partial J}\widetilde{P}_{k,k+1} = s_{n-k},$$
$$\frac{\partial}{\partial J}\widetilde{P}_{k,k-1} = s_k, \quad \text{and}$$
$$\frac{\partial}{\partial J}\widetilde{P}_{k,k} = -s_k - s_{n-k},$$

where k changes between 0 and n.

Let $\widetilde{P}'$ denote $\partial\widetilde{P}/\partial J$. Then we can write:

$$f_0(\widetilde{P}'f)_0 = f_0 s_n(f_1 - f_0),$$
$$f_k(\widetilde{P}'f)_k = f_k[-s_k(f_k - f_{k-1}) + s_{n-k}(f_{k+1} - f_k)], \quad \text{if } 1 \le k \le n-1,$$
$$f_n(\widetilde{P}'f)_n = f_n[-s_n(f_n - f_{n-1})].$$

Since the eigenvector f is increasing, hence all differences $f_k - f_{k-1}$ are non-negative. Moreover, if $k \le (n-1)/2$, then $f_k \le 0$, $s_k \ge 0$, and $s_{n-k} \le 0$, which implies that $f_k(\widetilde{P}'f)_k \ge 0$. Similarly, $k \ge (n+1)/2$ implies that $f_k(\widetilde{P}'f)_k \ge 0$. The only remaining case is when n is even and $k = n/2$. However, in this case $f_k = 0$ and therefore $f_k(\widetilde{P}'f)_k = 0$. It follows that

$$\left\langle f, \frac{\partial\widetilde{P}}{\partial J} f \right\rangle_\pi = \sum_{k=0}^{n} \pi_k f_k (\widetilde{P}'f)_k \ge 0.$$

This completes the proof of Theorem 1. □

References

Diaconis, P., 1996. The cutoff phenomenon in finite Markov chains. Proceedings of the National Academy of Sciences 93 (4), 1659–1664.

Ding, J., Lubetzky, E., Peres, Y., 2009. The mixing time evolution of Glauber dynamics for the mean-field Ising model. Communications in Mathematical Physics 289, 725–764.

Griffiths, R.B., Weng, C.-Y., Langer, J.S., 1966. Relaxation times for metastable states in the mean-field model of a ferromagnet. Physical Review 149, 301–305.

Levin, D.A., Luczak, M.J., Peres, Y., 2010. Glauber dynamics for the mean-field Ising model: cut-off, critical power law, and metastability. Probability Theory and Related Fields 146, 223–265.

Levin, D.A., Peres, Y., Wilmer, E.L., 2009. Markov Chains and Mixing Times. American Mathematical Society.

Nacu, S., 2003. Glauber dynamics on the cycle is monotone. Probability Theory and Related Fields 127, 177–185.