Int J Game Theory (2008) 37:211–218
DOI 10.1007/s00182-007-0106-1

ORIGINAL PAPER

On coordination games with quantum correlations

Vladislav Kargin

Accepted: 8 July 2007 / Published online: 3 August 2007

Abstract A necessary condition is derived that helps to determine whether an entangled quantum system can improve coordination in a game with incomplete information.

Keywords Quantum games · Coordination game · Correlated equilibrium

JEL Classification Numbers 91A28 · 91A99 · 81P68

In a recent surprising development, game theory has been applied to conflict situations in which outcomes depend both on participants' actions and on results of a quantum system measurement. These conflict situations have been named quantum games.[1] The potential usefulness of a quantum system stems from the fact that measurements of two remote quantum particles exhibit a correlated behavior that cannot be reproduced using classical correlation devices. Indeed, it turns out that in certain games the players can improve coordination of their actions using these quantum correlations. Here we derive a necessary condition for this to be possible.

Let us use the probability space (Ω, Σ, μ) that consists of the interval $\Omega = [0, 1]$, the sigma algebra Σ of Borel subsets in Ω, and the Borel–Lebesgue measure μ.

Definition 1 A *coordination game* with incomplete information, G, is defined by (i) random variables φ_A, φ_B, x_A, and x_B which take values in finite sets Φ_A, Φ_B, X_A, and X_B, (ii) finite sets $\mathfrak{A}_A$ and $\mathfrak{A}_B$, and (iii) a function U that maps $\Phi_A \times \Phi_B \times \mathfrak{A}_A \times \mathfrak{A}_B$

[1] See seminal papers by Meyer (1999) and Eisert et al. (1999), and further developments in Benjamin and Hayden (2001), Kay et al. (2001), Lee and Johnson (2003a,b), Landsburg (2004), Brassard et al. (2005), Chen and Hogg (2006), and Patel (2007).

The author would like to express his gratitude to Diana Bloom for her help with editing this paper.

V. Kargin (✉)
Courant Institute of Mathematical Sciences, 251 Mercer Street, New York, NY 10012, USA
e-mail: kargin@cims.nyu.edu

to real numbers. R.v. φ_A, φ_B are called *types* of players A and B, r.v. x_A, and x_B are called *signals* of players A and B, elements of $\mathfrak{A}_A$ and $\mathfrak{A}_B$ are called *actions* of players A and B, and U is called the *utility function*. The *strategies* of the game G are elements of finite sets $[\mathfrak{A}_A]^{\Phi_A \times X_A}$ and $[\mathfrak{A}_B]^{\Phi_B \times X_B}$. That is, a strategy of player A, s_A, is a map from $\Phi_A \times X_A$ to $\mathfrak{A}_A$. Similar, a strategy of player B, s_B, is a map from $\Phi_B \times X_B$ to $\mathfrak{A}_B$. The players' payoffs are the expected utilities if strategies s_A and s_B are played:

$$\pi_B(s_A, s_B) = \pi_A(s_A, s_B)$$
$$=: \int_\Omega U[s_A(\varphi_A(\omega), x_A(\omega)), s_B(\varphi_B(\omega), x_B(\omega)), \varphi_A(\omega), \varphi_B(\omega)]\, d\mu(\omega).$$

Note that we assume that the players' payoffs are equal in this game, so there is no conflict of interest between players. We will denote the payoff function as $\pi(s_A, s_B)$ (without subscript).

The interpretation of the game is that player A observes realizations of r.v. φ_A and x_A, player B observes realizations of r.v. φ_B and x_B, and then they choose actions from sets $\mathfrak{A}_A$ and $\mathfrak{A}_B$. Random variables φ_A and φ_B are external information provided to them, and x_A and x_B are information obtained by the measurement of a quantum system. The players' utilities from playing a_A and a_B are given by $u_A = u_B = U(a_A, a_B, \overline{\varphi}_A, \overline{\varphi}_B)$, where $\overline{\varphi}_A \in \Phi_A$, $\overline{\varphi}_B \in \Phi_B$, $a_A \in \mathfrak{A}_A$, and $a_B \in \mathfrak{A}_B$. (Throughout the paper a bar on top of a symbol means that this is a value of the r.v. corresponding to the symbol.) Note that the utilities and payoffs do not directly depend on the realization of signals x_A and x_B, so these r.v. only help the players to coordinate their strategies.

Using two variables to describe information available to a player is a departure from the standard model, in which a player's type consists of all information that the player gets (see, e.g., Harsanyi 1967 or Fudenberg and Tirole 1991). However, in our model random variables φ and x play quite distinct roles and it is convenient to have special names for them. Essentially, φ_A and φ_B describe information that players get externally. The generation of this information cannot be changed by them. In contrast, x_A and x_B are chosen by players. They are outcomes of a quantum system measurement, which can be chosen by players depending on realizations of φ_A and φ_B. We assume that the players agree on details of the measurement and therefore on the distribution of r.v. x_A and x_B before playing the game. The ability of players to choose the distribution of signals means that they are initially faced with a family of games, which have a fixed distribution of types but varying distributions of signals. The task of the players is to choose a game with the distribution of x_A and x_B that will provide them with the largest equilibrium payoff.

Signals x_A and x_B can be used for coordination of actions, communication of information about types, or both. Since signals x_A and x_B are results of a measurement of a physical system that exists at locations of both A and B, therefore, x_A and x_B can reflect the messages that players A and B each send to the other. For example, suppose A has at her disposal a communication channel that allows her to transmit any message to B. Suppose also that A and B have agreed that A will transmit a message about her type. We can model this communication scheme by postulating that $x_B = \varphi_A$. In terms of probability distributions, this communication channel is described by a probability distribution of types and signals that satisfies the property $P\{x_B \neq \varphi_A\} = 0$. The idea that all the properties of a communication channel can be described by the joint probability distribution of inputs and outputs goes back to Shannon (see Shannon and Weaver 1999). Of course, in our case the perfect communication channel is not available and we focus on a specific class of the probability distributions that could arise from a measurement of a quantum device.

To reiterate, the types φ_A and φ_B are inherent and cannot be changed by players. In contrast, the communication scheme and the corresponding distribution of signals x_A and x_B are at the disposal of the players subject to constraints imposed by physical properties of the communication channel. Once the communication scheme is fixed we have a game with incomplete information as defined above. However, the choice of the communication scheme will be modeled not as a game but as a decision problem that players must solve before starting the game. This formulation is possible because the players do not have a conflict of interest and seek to choose the communication scheme so as to maximize their common equilibrium payoff in the ensuing game.

Definition 2 A *Bayesian Nash equilibrium* of coordination game G is such a pair (s_A^*, s_B^*) that for all s_A, s_B, $\pi\left(s_A^*, s_B^*\right) \geq \pi\left(s_A^*, s_B\right)$ and $\pi\left(s_A^*, s_B^*\right) \geq \pi\left(s_A, s_B^*\right)$.

Lemma 1 *Let s_A^* and s_B^* be the strategies that maximize $\pi(s_A, s_B)$. Then (s_A^*, s_B^*) is a Bayesian Nash equilibrium of coordination game G.*

Proof is by direct verification.

Let $S_{\overline{\varphi}_A,\overline{\varphi}_B,\overline{x}_A,\overline{x}_B} \equiv \{\omega \in \Omega : \varphi_A(\omega) = \overline{\varphi}_A, \varphi_B(\omega) = \overline{\varphi}_B, x_A(\omega) = \overline{x}_A, x_B(\omega) = \overline{x}_B\}$.

Definition 3 The *distribution function* of the r.v. φ_A, φ_B, x_A, and x_B is a map from $\Phi_A \times \Phi_B \times X_A \times X_B$ to $\mathbb{R}$ defined by

$$p_{\varphi_A,\varphi_B,x_A,x_B}\left(\overline{\varphi}_A, \overline{\varphi}_B, \overline{x}_A, \overline{x}_B\right) =: \mu\left(S_{\overline{\varphi}_A,\overline{\varphi}_B,\overline{x}_A,\overline{x}_B}\right).$$

To lighten the notation, we write $p\left(\overline{\varphi}_A, \overline{\varphi}_B, \overline{x}_A, \overline{x}_B\right)$ instead of $p_{\varphi_A,\varphi_B,x_A,x_B}\left(\overline{\varphi}_A, \overline{\varphi}_B, \overline{x}_A, \overline{x}_B\right)$ if there is no ambiguity. We also use similar notation for conditional distribution functions. For example, let $S_{\overline{x}_B} = \{\omega \in \Omega : x_B(\omega) = \overline{x}_B\}$, then

$$p\left(\overline{\varphi}_A, \overline{\varphi}_B, \overline{x}_A | \overline{x}_B\right) =: \frac{\mu\left(S_{\overline{\varphi}_A,\overline{\varphi}_B,\overline{x}_A,\overline{x}_B}\right)}{\mu\left(S_{\overline{x}_B}\right).},$$

if $\mu\left(S_{\overline{x}_B}\right) > 0$, and $=: 0$ otherwise. Using the distribution function, we can write the payoff in the game G as a finite sum over values of r.v. $\varphi_A, \varphi_B, x_A, x_B$:

Lemma 2

$$\pi\left(s_A, s_B\right) = \sum_{\substack{\overline{\varphi}_A \in \Phi_A, \overline{\varphi}_B \in \Phi_B, \\ \overline{x}_A \in X_A, \overline{x}_B \in X_B}} U\left(s_A\left(\overline{\varphi}_A, \overline{x}_A\right), s_B\left(\overline{\varphi}_B, \overline{x}_B\right), \overline{\varphi}_A, \overline{\varphi}_B\right) \times p\left(\overline{\varphi}_A, \overline{\varphi}_B, \overline{x}_A, \overline{x}_B\right). \tag{1}$$

Proof is by direct verification.

Definition 4 Distribution of types and signals $p\left(\overline{\varphi}_A, \overline{\varphi}_B, \overline{x}_A, \overline{x}_B\right)$ is called *consistent* with distribution of types $p\left(\overline{\varphi}_A, \overline{\varphi}_B\right)$ if the φ_A-φ_B marginal of $p\left(\overline{\varphi}_A, \overline{\varphi}_B, \overline{x}_A, \overline{x}_B\right)$ coincides with $p\left(\overline{\varphi}_A, \overline{\varphi}_B\right)$:

$$\sum_{\overline{x}_A \in X_A, \overline{x}_B \in X_B} p\left(\overline{\varphi}_A, \overline{\varphi}_B, \overline{x}_A, \overline{x}_B\right) = p\left(\overline{\varphi}_A, \overline{\varphi}_B\right)$$

for every $\overline{\varphi}_A \in \Phi_A$ and $\overline{\varphi}_B \in \Phi_B$.

Definition 5 Games G and G' are *similar* if they have the same type and strategy sets $\Phi_A, \Phi_B, \mathfrak{A}_A$ and $\mathfrak{A}_B$, utility function $U\left(a_A, a_B, \overline{\varphi}_A, \overline{\varphi}_B\right)$, and probability space (Ω, Σ, μ), and if their signals and types are such that distributions $p\left(\overline{\varphi}_A, \overline{\varphi}_B, \overline{x}_A, \overline{x}_B\right)$ and $p'\left(\overline{\varphi}_A, \overline{\varphi}_B, \overline{x}_A, \overline{x}_B\right)$ are consistent with the same distribution of types, $p\left(\overline{\varphi}_A, \overline{\varphi}_B\right)$.

Intuitively, before playing, players are free to choose a communication/coordination scheme and we model this freedom by allowing the players to choose the distribution $p\left(\overline{\varphi}_A, \overline{\varphi}_B, \overline{x}_A, \overline{x}_B\right)$. However, this choice is restricted: the players can choose the distribution of their signals but not the distribution of their types. We will impose additional restriction on the distribution of signals below. Similarity is evidently an equivalence relation on the set of coordination games.

Definition 6 R.v. φ_A, φ_B, x_A, and x_B and their distribution $p\left(\overline{\varphi}_A, \overline{\varphi}_B, \overline{x}_A, \overline{x}_B\right)$ are called *disjoint* if

$$\begin{aligned} p\left(\overline{\varphi}_B | \overline{\varphi}_A, \overline{x}_A\right) &= p\left(\overline{\varphi}_B | \overline{\varphi}_A\right), \\ p\left(\overline{\varphi}_A | \overline{\varphi}_B, \overline{x}_B\right) &= p\left(\overline{\varphi}_A | \overline{\varphi}_B\right). \end{aligned} \tag{2}$$

In other words, signal x_A does not provide any additional information about type φ_B which is not already in type φ_A, and similarly for signal x_B. We can say informally that signal x_A is useless for communication of information about φ_B. Indeed, any message from player B that could possibly be coded in signal x_A does not change the probability distribution of φ_B that player A infers based on the realization of her type φ_A only. Similarly, x_B is useless for communication of information about φ_A.

Definition 7 R.v. $\varphi_A, \varphi_B, x_A$, and x_B, and their distribution are *classically generated* if there exists a r.v., u, independent from φ_A and φ_B such that the following equality holds:

$$p(\overline{x}_A, \overline{x}_B | \overline{u}, \overline{\varphi}_A, \overline{\varphi}_B) = p(\overline{x}_A | \overline{u}, \overline{\varphi}_A) p(\overline{x}_B | \overline{u}, \overline{\varphi}_B). \tag{3}$$

For example, the r.v. $\varphi_A, \varphi_B, x_A$, and x_B are classically generated if there exists a r.v. u, independent of φ_A and φ_B, and functions $\widetilde{x}_A$ and $\widetilde{x}_B$, such that $x_A = \widetilde{x}_A(\varphi_A, u)$ and $x_B = \widetilde{x}_B(\varphi_B, u)$. Informally, both players have access to a coordination device but do not attempt to communicate: they can only observe a random signal u, which is independent of their types. Clearly, classically generated r.v. are disjoint.

So far we have not mentioned quantum mechanics. Here is the piece whose interest depends on the existence of certain long-range correlations discovered in physics.

Definition 8 R.v. $\varphi_A, \varphi_B, x_A$, and x_B are called *entangled* if they are disjoint and are not classically generated.

The famous non-locality theorem by Bell (see Clauser et al. 1969) says that outcomes of certain measurements performed on two remote parts of a quantum system can be entangled in the sense of our definition. The outcomes of the measurements correspond to our signals and the positions of the measurement apparatuses to the types. The existence of non-classical correlations has been confirmed in experiments.

Initially, the existence of entangled measurements raised the concern that it violates the Einstein postulate that no physical action can propagate faster than the speed of light. However, it was soon discovered that the entangled measurements cannot be used to transmit information, and this can be seen as evidence that the Einstein postulate is not violated. We capture the existence of the measurements with non-classical correlations by the concept of entangled signals. It is worth noting, however, that not every quadruple of signals that is entangled in our sense can be realized by measurements of a quantum system (Cirel'son 1980). For some properties of entangled signals see Barret et al. (2005), where they are called non-local correlations.

It is surprising but the entangled signals—although useless for communication of information—can be useful in a coordination game. An example by Cleve et al. (2004) shows that measurements of a quantum system can increase the game payoff relative to the case when only classically generated variables are available. This example is effectively a representation of inequality by Clauser et al. (1969) in a game theoretic setting. Does sharing a quantum system always increase the expected payoff in coordination games? No. A necessary condition is that the utility function depend on both players' types.

Theorem 1 *For every game G with independent types, disjoint signals and the utility function that depends only on the first player's type: $U = U(a, b; \varphi_A)$, and for every Bayesian Nash equilibrium $\left(s_A^{(G)}, s_B^{(G)}\right)$ of G, there exists a similar game H with classically generated signals and a Bayesian Nash equilibrium $\left(s_A^{(H)}, s_B^{(H)}\right)$ of H such that $\pi^{(H)}\left(s_A^{(H)}, s_B^{(H)}\right) \geq \pi^{(G)}\left(s_A^{(G)}, s_B^{(G)}\right)$ where $\pi^{(H)}$ and $\pi^{(G)}$ are the payoff functions of H and G, respectively.*

Proof Let s_A^* and s_B^* be the players' strategies that maximize the payoff $\pi^{(G)}(s_A, s_B)$ in game G. Then evidently $\pi^{(G)}\left(s_A^*, s_B^*\right) \geq \pi^{(G)}\left(s_A^{(G)}, s_B^{(G)}\right)$ for any equilibrium $\left(s_A^{(G)}, s_B^{(G)}\right)$ of G. Next, note that $x'_A =: s_A^*(x_A, \varphi_A)$, and $x'_B =: s_B^*(x_B, \varphi_B)$ are disjoint r.v., and that distribution $p\left(\overline{\varphi}_A, \overline{\varphi}_B, \overline{x}'_A, \overline{x}'_B\right)$ is consistent with $p\left(\overline{\varphi}_A, \overline{\varphi}_B\right)$. Define game G' with the same probability space, type and strategy sets, utility function, and even the same types as G but with signals x'_A, and x'_B instead of x_A and x_B. Game G' has the following properties: (i) G' is similar to G; (ii) G' is a game with disjoint signals; (iii) signals of G' take values in the sets of strategies: $X'_A \subseteq \mathfrak{A}_A$ and $X'_B \subseteq \mathfrak{A}_B$; (iv) $\pi^{(G')}\left(s'_A, s'_B\right) = \pi^{(G)}\left(s_A^*, s_B^*\right)$, where s'_A and s'_B are the strategies of G' defined as $s'_A(\overline{\varphi}_A, \overline{x}'_A) = \overline{x}'_A$ and $s'_B(\overline{\varphi}_B, \overline{x}'_B) = \overline{x}'_B$. Suppose for the moment that we have proved that for each game G' with these properties, there exists a game G'' with classically generated signals and such that (a) G'' is similar to G'; (b) signals of G'' take values in the sets of strategies: $X''_A \subseteq \mathfrak{A}_A$ and $X''_B \subseteq \mathfrak{A}_B$, and (c) $\pi^{(G'')}\left(s''_A, s''_B\right) = \pi^{(G')}\left(s'_A, s'_B\right)$, where s''_A and s''_B are the strategies of G'' defined as $s''_A(\overline{\varphi}_A, \overline{x}''_A) = \overline{x}''_A$ and $s''_B(\overline{\varphi}_B, \overline{x}''_B) = \overline{x}''_B$. Then by Lemma 1, there is a Bayesian Nash equilibrium in G'', say $\left(s^{**}_A, s^{**}_B\right)$, such that

$$\pi^{(G'')}\left(s^{**}_A, s^{**}_B\right) \geq \pi^{(G'')}\left(s''_A, s''_B\right) = \pi^{(G')}\left(s'_A, s'_B\right) = \pi^{(G)}\left(s^{*}_A, s^{*}_B\right)$$

Therefore, we can take $H = G''$ and the statement of the theorem will be valid.

Now let us prove the existence of the game G'' assumed above. Let the joint distribution of types and signals in game G' be $p'(\overline{\varphi}_A, \overline{\varphi}_B, \overline{x}'_A, \overline{x}'_B)$. Consider the following distribution function on $\Phi_A \times \Phi_B \times X'_A \times X'_B$:

$$p''(\overline{\varphi}_A, \overline{\varphi}_B, \overline{x}_A, \overline{x}_B) = p'(\overline{\varphi}_A, \overline{x}_A, \overline{x}_B) p'(\overline{\varphi}_B). \tag{4}$$

We have to prove the existence of r.v. x''_A, x''_B, φ''_A, and φ''_B with this distribution function. (In (4) $\overline{x}_A, \overline{x}_B, \overline{\varphi}_A$, and $\overline{\varphi}_B$ denote the general elements of the sets $X'_A \times X'_B \times \Phi_A \times \Phi_B$; they denote values of r.v. x'_A, x'_B, φ_A, and φ_B if used as arguments of the distribution p', and values of x''_A, x''_B, φ''_A, and φ''_B if used as arguments of distribution p''. We will also use this convention in the following.) Consider the probability space defined by the Borel–Lebesgue measure on the square $\Omega \times \Omega$. Define r.v. $\widetilde{x}_A$, $\widetilde{x}_B$, and $\widetilde{\varphi}_A$ by the formulas $\widetilde{x}_A(\omega_1, \omega_2) = x'_A(\omega_1)$, $\widetilde{x}_B(\omega_1, \omega_2) = x'_B(\omega_1)$, and $\widetilde{\varphi}_A(\omega_1, \omega_2) = \varphi_A(\omega_1)$, where ω_1 and ω_2 are coordinates of a point in $\Omega \times \Omega$. Define r.v. $\widetilde{\varphi}_B$ by the formula $\widetilde{\varphi}_B(\omega_1, \omega_2) = f(\omega_2)$ where f is a measurable function that takes values in the set Φ_B and has the distribution function $p'(\overline{\varphi}_B)$. (The existence of such a function is clear.) Then r.v. $\widetilde{x}_A$, $\widetilde{x}_B$, $\widetilde{\varphi}_A$ and $\widetilde{\varphi}_B$ have the distribution function p'' from (4). Next use the theorem about the measure-theoretical isomorphism of all Lebesgue measure spaces (see, e.g., Rohlin 1962). Take a map U that establishes the isomorphism between Ω and $\Omega \times \Omega$ and define $x''_A(\omega) = \widetilde{x}_A(U\omega)$, $x''_B(\omega) = \widetilde{x}_B(U\omega)$, $\varphi''_A(\omega) = \widetilde{\varphi}_A(U\omega)$, and $\varphi''_B(\omega) = \widetilde{\varphi}_B(U\omega)$. Then r.v. x''_A, x''_B, φ''_A, and φ''_B are defined on Ω, take values in X'_A, X'_B, Φ_A, and Φ_B, respectively, and have the distribution function p'' from (4).

Distribution p'' is consistent. Indeed

$$\begin{aligned}
p''(\overline{\varphi}_A, \overline{\varphi}_B) &= \sum_{\overline{x}_A \in X'_A, \overline{x}_B \in X'_B} p''(\overline{x}_A, \overline{x}_B, \overline{\varphi}_A, \overline{\varphi}_B) \\
&= \sum_{\overline{x}_A \in X'_A, \overline{x}_B \in X'_B} p'(\overline{x}_A, \overline{x}_B, \overline{\varphi}_A) p'(\overline{\varphi}_B) \\
&= \sum_{\overline{x}_A \in X'_A, \overline{x}_B \in X'_B} \left(\sum_{\overline{\varphi}_B \in \Phi_B} p'(\overline{x}_A, \overline{x}_B, \overline{\varphi}_A, \overline{\varphi}_B) \right) p'(\overline{\varphi}_B) \\
&= \sum_{\overline{x}_A \in X'_A, \overline{x}_B \in X'_B} \left(\sum_{\overline{\varphi}_B \in \Phi_B} p'(\overline{x}_A, \overline{x}_B | \overline{\varphi}_A, \overline{\varphi}_B) p'(\overline{\varphi}_A) p'(\overline{\varphi}_B) \right) p'(\overline{\varphi}_B)
\end{aligned}$$

$$= \sum_{\overline{x}_A \in X'_A, \overline{x}_B \in X'_B} p'(\overline{x}_A, \overline{x}_B | \overline{\varphi}_A, \overline{\varphi}_B) p'(\overline{\varphi}_A) p'(\overline{\varphi}_B)$$
$$= p'(\overline{\varphi}_A, \overline{\varphi}_B)$$

Note that here we have substantially used the assumption that φ_A and φ_B are independent.

Define game G'' as a game similar to game G', in which the signals and types are x''_A, x''_B, φ''_A, and φ''_B. Then $\pi^{(G'')}\left(s''_A, s''_B\right) = \pi^{(G')}\left(s'_A, s'_B\right)$. Indeed,

$$\pi^{(G'')}\left(s''_A, s''_B\right) = \sum_{\substack{\overline{x}_A \in X'_A, \overline{x}_B \in X'_B, \\ \overline{\varphi}_A \in \Phi_A, \overline{\varphi}_B \in \Phi_B}} U(\overline{x}_A, \overline{x}_B; \overline{\varphi}_A) p''(\overline{x}_A, \overline{x}_B, \overline{\varphi}_A, \overline{\varphi}_B) \tag{5}$$

$$= \sum_{\substack{\overline{x}_A \in X'_A, \overline{x}_B \in X'_B, \\ \overline{\varphi}_A \in \Phi_A}} U(\overline{x}_A, \overline{x}_B; \overline{\varphi}_A) p'(\overline{x}_A, \overline{x}_B, \overline{\varphi}_A) \tag{6}$$

$$= \pi^{(G')}\left(s'_A, s'_B\right) \tag{7}$$

We will complete the proof of the theorem by showing that signals with distribution $p''(\overline{x}_A, \overline{x}_B, \overline{\varphi}_A, \overline{\varphi}_B)$ can be classically generated. First, note that if distribution $p'(\overline{x}_A, \overline{x}_B, \overline{\varphi}_A, \overline{\varphi}_B)$ is disjoint and φ_A is independent from φ_B, then φ_A and x'_B are also independent. Indeed, by the definition of disjoint signals (2) and the independence of φ_A and φ_B, we have $p'(\overline{\varphi}_A | \overline{\varphi}_B, \overline{x}_B) = p'(\overline{\varphi}_A | \overline{\varphi}_B) = p'(\overline{\varphi}_A)$. Consequently,

$$p'(\overline{\varphi}_A, \overline{x}_B) = \sum_{\overline{\varphi}_B \in \Phi_B} p'(\overline{\varphi}_A | \overline{\varphi}_B, \overline{x}_B) p'(\overline{\varphi}_B, \overline{x}_B) = p'(\overline{\varphi}_A) \sum_{\overline{\varphi}_B \in \Phi_B} p'(\overline{\varphi}_B, \overline{x}_B)$$
$$= p'(\overline{\varphi}_A) p'(\overline{x}_B). \tag{8}$$

Second, by definition of p'' and (8): $p''(\overline{x}_B, \overline{\varphi}_A, \overline{\varphi}_B) = p'(\overline{x}_B, \overline{\varphi}_A) p'(\overline{\varphi}_B) = p'(\overline{x}_B) p'(\overline{\varphi}_A) p'(\overline{\varphi}_B) = p''(\overline{x}_B) p''(\overline{\varphi}_A) p''(\overline{\varphi}_B)$. That is, x''_B is independent of both φ''_A and φ''_B under distribution p''. Using the fact that φ''_B is independent of everything else under p'', and the identity $p''(\overline{x}_B | \overline{x}_B, \overline{\varphi}_B) = 1$, we obtain

$$p''(\overline{x}_A, \overline{x}_B | \overline{x}_B, \overline{\varphi}_A, \overline{\varphi}_B) = p''(\overline{x}_A | \overline{x}_B, \overline{\varphi}_A) p''(\overline{x}_B | \overline{x}_B, \overline{\varphi}_B), \tag{9}$$

as required by the definition of the classically generated variables (3). □

Remark The author does not know if the assumption of independence of φ_A and φ_B can be relaxed in the above theorem. Also, the dependence of payoff on both types is clearly not sufficient to ensure that entangled signals can improve over classically generated signals. For example, if there is a pair of actions that gives the maximal payoff for any combination of types, then there clearly cannot be any improvement obtained by using the entangled signals.

The result of the theorem can be explained in the following intuitive way. By the condition of disjointness, a player's signal does not inform him about the other player's type. Moreover, the entangled signals do not provide a player with better information about the other player's signal than information that could be achieved with classical signals. The entangled signals can still provide some improved information about the joint distribution of the type and the action of the other player relative to classical signals. However, if the payoff function does not depend on both the type and the action of the other player, then this additional information has no value.

References

Barret J, Linden N, Massar S, Pironio S, Popescu S, Roberts D (2005) Non-local correlation as an information theoretic resource. Phys Rev A 71:022101

Benjamin SC, Hayden PM (2001) Multiplayer quantum games. Phys Rev A 64:030301

Brassard G, Broadbent A, Tapp A (2005) Quantum pseudo-telepathy. Found Phys 35(11):1877–1907

Chen K-Y, Hogg T (2006) How well do people play a quantum prisoner's dilemma. Quantum Inf Process 5:43–67

Cirel'son B (1980) Quantum generalizations of Bell's inequality. Lett Math Phys 4:93–100

Clauser JF, Horne MA, Shimony A, Holt RA (1969) Proposed experiment to test local hidden-variable theories. Phys Rev Lett 23:880–884

Cleve R, Hoyer P, Toner B, Watrous J (2004) Consequences and limits of nonlocal strategies. Available at http://arxiv.org/abs/quant-ph/0404076

Eisert J, Wilkens M, Lewenstein M (1999) Quantum games and quantum strategies. Phys Rev Lett 83: 3077–3080

Fudenberg D, Tirole J (1991) Game theory. MIT Press, Cambridge

Harsanyi JC (1967) Games with incomplete information played by Bayesian players, I and II. Manag Sci 14:159–182, 320–334

Kay R, Johnson NF, Benjamin SC (2001) Evolutionary quantum game. J Phys A 34:L547–L552

Landsburg SE (2004) Quantum game theory. Not AMS 51:394–399

Lee CF, Johnson NF (2003a) Efficiency and formalism of quantum games. Phys Rev A 67:022311

Lee CF, Johnson NF (2003b) Game-theoretic discussion of quantum state estimation and cloning. Phys Lett A 319:429–433

Meyer DA (1999) Quantum strategies. Phys Rev Lett 82:1052–1055

Patel N (2007) States of play. Nature 445:144–146

Rohlin VA (1962) On the fundamental ideas of measure theory. Translations of American Mathematical Society, Series 1, vol 10. American Mathematical Society, Providence, Rhode Island

Shannon CE, Weaver W (1999) Mathematical theory of communication. University of Illinois Press. Reprint of 1949 edition