

ON SUPERCONVERGENCE OF SUMS OF FREE RANDOM VARIABLES

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This paper derives sufficient conditions for superconvergence of sums of bounded free random variables and provides an estimate for the rate of superconvergence.

1. Introduction. Free probability theory is an interesting generalization of classical probability theory to a noncommutative setting. It was introduced in the mid-1980's by Voiculescu [18–20] as a tool for studying type II₁ von Neumann algebras. In many respects, free probability theory parallels classical probability theory. There exist analogues of the central limit theorem [19], the law of large numbers [6] and the classification of infinitely divisible and stable laws [3, 7]. On the other hand, certain features of free and classical probability theories differ strikingly. Let $S_n = n^{-1/2} \sum_{i=1}^n X_i$, where X_i are identically distributed and free random variables. Then the law of S_n approaches the limit law in a completely different manner than in the classical case. To illustrate this, suppose that the support of X_i is $[-1, 1]$. Take a positive number $\alpha < 1$. Then, in the classical case, the probability of $\{|S_n| > \alpha n\}$ is exponentially small, but not zero. In contrast, in the noncommutative case, the probability becomes identically zero for all sufficiently large n . This mode of convergence is called *superconvergence* in [5].

In this paper, we extend the superconvergence result to a more general setting of nonidentically distributed variables and estimate the rate of superconvergence quantitatively. It turns out, in particular, that the support of S_n can deviate from the supporting interval of the limiting law by not more than c/\sqrt{n} and we explicitly estimate the constant c . An example shows that the rate $n^{-1/2}$ in this estimate cannot be improved.

Related results have been obtained in the random matrix literature. For example, [10] considers the distribution of the largest eigenvalue of an empirical covariance matrix for a sample of Gaussian vectors. This problem can be seen as a problem concerning the edge of the spectrum of a sum of n random rank-one operators in the N -dimensional vector space. More precisely, the question concerns sums of the form $S_n = \sum_{i=1}^n x_i x_i'$, where x_i is a random N -vector with entries distributed

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according to the Gaussian law with the normalized variance $1/N$. Then S_n is a matrix-valued random variable with the Wishart distribution.

Johnstone is interested in the asymptotic behavior of the distribution of the largest eigenvalue of S_n . The asymptotics are derived under the assumptions that both n and N approach ∞ and that $\lim n/N = \gamma > 0$, $\gamma \neq \infty$. Johnstone found that the largest eigenvalue has variance of the order $n^{-2/3}$ and that after an appropriate normalization, the distribution of the largest eigenvalue approaches the Tracy–Widom law. This law has a right-tail asymptotically equivalent to $\exp[-(2/3)s^{3/2}]$ and, in particular, is unbounded from above. Johnstone’s results have generalized the original breakthrough results in [16] (see also [17]) for self-adjoint random matrices without covariance structure. In [14] and [15], it is shown that the results regarding the asymptotic distribution of the largest eigenvalue remain valid even if the matrix entries are not necessarily Gaussian.

An earlier contribution, [2], also considered empirical covariance matrices of large random vectors that are not necessarily Gaussian and studied their largest eigenvalues. Again, both n and N approach infinity and $\lim n/N = \gamma > 0$, $\gamma \neq \infty$. In contrast to Johnstone, Bai and Silverstein were interested in the behavior of the largest eigenvalue along a sequence of increasing random covariance matrices. Suppose that the support of the limiting eigenvalue distribution is contained in the interior of a closed interval, I . Bai and Silverstein showed that the probability that the largest eigenvalue lies outside I is zero for all sufficiently large n .

These results are not directly comparable with ours for several reasons. First, in our case, the edge of the spectrum is not random in the classical sense and so it does not make sense to talk about its variance. Second, informally speaking, we are looking at the limit situation when $N = \infty$, $n \rightarrow \infty$. Because of this, we use much easier techniques than all of the aforementioned papers, as we do not need to handle the interaction of the randomness and the passage to the asymptotic limit. Despite these differences, comparison of our results with the results of the random matrix literature is stimulating. In particular, superconvergence in free probability theory can be thought as an analogue of the Bai–Silverstein result.

The rest of the paper is organized as follows. Section 2 provides the necessary background about free probability theory and describes the main result, Section 3 recalls some results that we will need in the proof and Section 4 is devoted to the proof of the main result.

2. Main theorem.

DEFINITION 1. A *noncommutative probability space* is a pair (\mathcal{A}, E) , where \mathcal{A} is a unital C^* -algebra of bounded linear operators acting on a complex separable Hilbert space and E is a linear functional from \mathcal{A} to complex numbers. The operators belonging to the algebra \mathcal{A} are called *noncommutative random variables* or simply *random variables* and the functional E is called the *expectation*.

An algebra of bounded linear operators is a unital C^* -algebra if it contains the identity operator I and if it is closed with respect to the $*$ -operation, that is, if $A \in \mathcal{A}$, then $A^* \in \mathcal{A}$, where A^* is the adjoint of operator A . The algebra is also assumed to be closed with respect to convergence in the operator norm. The definition of noncommutative random variables can be generalized to include unbounded linear operators affiliated with algebra \mathcal{A} ; for details, see [4, 11]. In this paper, we restrict our attention to bounded random variables.

The linear functional E is assumed to satisfy the following properties (in addition to linearity): (i) $E(I) = 1$; (ii) $E(A^*) = E(A)$; (iii) $E(AA^*) \geq 0$; (iv) $E(AB) = E(BA)$; (v) $E(AA^*) = 0$ implies $A = 0$; and (vi) if $A_n \rightarrow A$, then $E(A_n) \rightarrow E(A)$.

For each self-adjoint operator A , the expectation induces a continuous linear functional on the space of continuous functions, $E_A : f \rightarrow Ef(A)$, and by the Riesz theorem, we can write this functional as a Stieltjes' integral of f over a measure. We call this measure, μ , the *measure associated with operator A and expectation E* . If $P(d\lambda)$ is the spectral resolution associated with operator A , then $\mu(d\lambda) = E(P(d\lambda))$. It is easy to check that μ is a probability measure on \mathbb{R} . If A is a bounded operator and $\|A\| \leq L$, then the support of μ is contained in the circle $|\lambda| \leq L$.

The most important concept in free probability theory is that of free independence of noncommuting random variables. Let a set of r.v.'s A_1, \dots, A_n be given. With each of them, we can associate an algebra \mathcal{A}_i , which is generated by A_i ; that is, it is the closure of all polynomials in variables A_i and A_i^* . Let \bar{A}_i denote an arbitrary element of algebra \mathcal{A}_i .

DEFINITION 2. The algebras $\mathcal{A}_1, \dots, \mathcal{A}_n$ (and variables A_1, \dots, A_n that generate them) are said to be *freely independent* or *free* if the following condition holds:

$$\varphi(\bar{A}_{i(1)}, \dots, \bar{A}_{i(m)}) = 0,$$

provided that $\varphi(\bar{A}_{i(s)}) = 0$ and $i(s + 1) \neq i(s)$.

In classical probability theory, one of the most important theorems is the central limit theorem (CLT). It has an analogue in noncommutative probability theory.

PROPOSITION 1. Let r.v.'s $X_i, i = 1, 2, \dots$, be self-adjoint and free. Assume that $E(X_i) = 0, \|X_i\| \leq L$ and $\lim_{n \rightarrow \infty} [E(X_1^2) + \dots + E(X_n^2)]/n = a_2$. Then measures associated with r.v.'s $n^{-1/2} \sum_{i=1}^n X_i$ converge in distribution to an absolutely continuous measure with density

$$\phi(x) = \frac{1}{2\pi a_2} \sqrt{4a_2 - x^2} \chi_{[-2\sqrt{a_2}, 2\sqrt{a_2}]}(x).$$

This result was proven in [18] and later generalized in [11] to unbounded identically distributed variables that have a finite second moment. Other generalizations can be found in [13] and [21].

In the classical case, the behavior of large deviations from the CLT is described by the Cramér theorem, the Bernstein inequality and their generalizations. It turns out that in the noncommutative case, the behavior of large deviations is considerably different. The theorem below gives some quantitative bounds on how the distribution of a sum of free random variables differs from the limiting distribution.

Let $X_{n,i}, i = 1, \dots, k_n$, be a double-indexed array of bounded self-adjoint random variables. The elements of each row, $X_{n,1}, \dots, X_{n,k_n}$, are assumed to be free, but are not necessarily identically distributed. Their associated probability measures are denoted $\mu_{n,i}$, their Cauchy transforms are $G_{n,i}(z)$, their k th moments are $a_{n,i}^{(k)}$, and so on. We define $S_n = X_{n,1} + \dots + X_{n,k_n}$ and the probability measure μ_n as the spectral probability measure of S_n . We are interested in the behavior of probability measure μ_n as n grows.

We will assume that the first moments of the random variables $X_{n,i}$ are zero and that $\|X_{n,i}\| \leq L_{n,i}$. Let $v_n = a_{n,1}^{(2)} + \dots + a_{n,k_n}^{(2)}$, $L_n = \max_i \{L_{n,i}\}$ and $T_n = L_{n,1}^3 + \dots + L_{n,k_n}^3$.

THEOREM 1. *Suppose that $\limsup_{n \rightarrow \infty} T_n/v_n^{3/2} < 2^{-12}$. Then for all sufficiently large n , the support of μ_n belongs to*

$$I = (-2\sqrt{v_n} - cT_n/v_n, 2\sqrt{v_n} + cT_n/v_n),$$

where $c > 0$ is an absolute constant.

REMARK 1. $c = 5$ will suffice although it is not the best possible.

REMARK 2. Informally, the assumption that $\limsup_{n \rightarrow \infty} T_n/v_n^{3/2} < 2^{-12}$ means that there are no large outliers. An example of when the assumption is violated is provided by random variables with variance $a_{n,i}^{(2)} = n^{-1}$ and $L_{n,i} = 1$. Then $T_n = n$ and $v_n^{3/2} = 1$, so that $T_n/v_n^{3/2}$ increases when n grows.

REMARK 3. The assumptions in Theorem 1 are weaker than the assumptions in Theorem 7 of [5]. In particular, Theorem 1 allows us to draw conclusions about random variables with nonuniformly bounded support. Consider, for example, random variables $X_k, k = 1, \dots, n$, that are supported on intervals $[-k^{1/3}, k^{1/3}]$ and have variances of order $k^{2/3}$. Then T_n has the order of n^2 and v_n has the order of $n^{5/3}$. Therefore, $T_n/v_n^{3/2}$ has the order of $n^{-1/2}$ and Theorem 1 is applicable. It allows us to conclude that the support of $S_n = X_1 + \dots + X_n$ is contained in the interval $(-2\sqrt{v_n} - cn^{1/3}, 2\sqrt{v_n} + cn^{1/3})$.

EXAMPLE 1 (*Identically distributed variables*). A particular case of the above scheme involves the normalized sums of identically distributed, bounded, free r.v.'s $S_n = (\xi_1 + \dots + \xi_n)/\sqrt{n}$. If $\|\xi_i\| \leq L$, then $\|\xi_i/\sqrt{n}\| \leq L_{n,i} = L_n = L/\sqrt{n}$. Therefore, $T_n = L^3/\sqrt{n}$. If the second moment of ξ_i is σ^2 , then the second moment of the sum S_n is $v_n = \sigma^2$. Applying the theorem, we obtain the result that starting with certain n , the support of the distribution of S_n belongs to $(-2\sigma - c(L^3/\sigma^2)n^{-1/2}, 2\sigma + c(L^3/\sigma^2)n^{-1/2})$.

EXAMPLE 2 (*Free Poisson*). Let the n th row of our scheme have $k_n = n$ identically distributed random variables $X_{n,i}$ with the Bernoulli distribution that places probability $p_{n,i}$ on 1 and $q_{n,i} = 1 - p_{n,i}$ on 0. (It is easy to normalize this distribution to have the zero mean by subtracting $p_{n,i}$.) Suppose that $\max_i p_{n,i} \rightarrow 0$ as $n \rightarrow \infty$ and that

$$\sum_{i=1}^n p_{n,i} \rightarrow \lambda > 0$$

as $n \rightarrow \infty$. Then $L_{n,i} \sim 1$ and $a_{n,i}^2 = p_{n,i}(1 - p_{n,i})$ so that $T_n \sim n$ and $v_n \rightarrow \lambda$ as $n \rightarrow \infty$. Therefore, Theorem 1 does not apply. An easy calculation for the case $p_{n,i} = \lambda/n$ shows that superconvergence still holds. This example shows that the conditions of the theorem are not necessary for superconvergence to hold.

EXAMPLE 3 (*Identically distributed binomial variables*). Let X_i be identically distributed with a distribution that attributes positive weights p and q to $-\sqrt{q/p}$ and $\sqrt{p/q}$, respectively. Then $EX_i = 0$ and $EX_i^2 = 1$. It is not difficult to show that the support of $S_n = n^{-1/2} \sum_{i=1}^n X_i$ is the interval $I = [x_1, x_2]$, where

$$x_{1,2} = \pm 2\sqrt{1 - \frac{1}{n} + \frac{q-p}{\sqrt{pq}} \frac{1}{\sqrt{n}}}.$$

This example shows that the rate of $n^{-1/2}$ in Theorem 1 cannot be improved without further restrictions. Note, also, that for $p > q$, L_n is $\sqrt{p/q}$ and therefore the coefficient preceding $n^{-1/2}$ is of order L_n . In the general bound, the coefficient is L_n^3/σ^2 . It is not clear whether it is possible to replace the coefficient in the general bound by a term of order L_n .

3. Preliminary results.

DEFINITION 3. The function

$$(1) \quad G(z) = \int_{-\infty}^{\infty} \frac{\mu(dt)}{z-t}$$

is called the *Cauchy transform* of the probability measure $\mu(dt)$.

The Cauchy transform encodes a wealth of information about the underlying probability measure. For our purposes, we need only some of them. First, the following inversion formula holds.

PROPOSITION 2 (The Stieltjes–Perron inversion formula). *For any interval $[a, b]$,*

$$\mu[a, b] = -\lim_{\varepsilon \downarrow 0} \frac{1}{\pi} \int_a^b \operatorname{Im} G(x + i\varepsilon) dx,$$

provided that $\mu(a) = \mu(b) = 0$.

A proof can be found in [1], pages 124–125.

We will call a function *holomorphic* at a point z if it can be represented by a convergent power series in a sufficiently small disc with center z . We call the function *holomorphic in an open domain* D if it is holomorphic at every point of the domain. Here, D may include $\{\infty\}$, in which case it is a part of the extended complex plane $\mathbb{C} \cup \{\infty\}$ with the topology induced by the stereographic projection of the Riemann sphere onto the extended complex plane.

The integral representation (1) shows that the Cauchy transform of every probability measure, $G(z)$, is a holomorphic function in $C^+ = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$ and $C^- = \{z \in \mathbb{C} \mid \operatorname{Im} z < 0\}$. If, in addition, the measure is assumed to be supported on interval $[-L, L]$, then the Cauchy transform is holomorphic in the area $\Omega : |z| > L$ where it can be represented by a convergent power series in z^{-1} ,

$$(2) \quad G(z) = \frac{1}{z} + \frac{m_1}{z^2} + \frac{m_2}{z^3} + \dots$$

Here, m_k denote the moments of the measure μ :

$$m_k = \int_{-\infty}^{\infty} t^k \mu(dt).$$

In particular, $G(z)$ is holomorphic at $\{\infty\}$. We call series (2) the G -series.

In the other direction, we have the following result.

LEMMA 1. *Suppose that:*

(1) $G(z)$ is the Cauchy transform of a compactly supported probability distribution, μ , and

(2) $G(z)$ is holomorphic at every $z \in \mathbb{R}$, $|z| > L$.

Then the support of μ lies entirely in the interval $[-L, L]$.

PROOF. From assumption (1), we infer that in some neighborhood of infinity, $G(z)$ can be represented by the convergent power series (2) and that $G(z)$ is also holomorphic everywhere in C^+ and C^- . Therefore, using assumption (2), we

can conclude that $G(z)$ is holomorphic everywhere in the area $\Omega = \{z \mid |z| > L\}$ including the point at infinity.

Let us detail the proof of this statement. Define

$$a = \inf\{l \geq 0 \mid G(z) \text{ is holomorphic on } |z| > l\}$$

and suppose, by seeking a contradiction, that $a > L$. Let ε be such that $a - \varepsilon > L$. Consider the area $\Omega_\varepsilon = \{z \mid a - \varepsilon < |z| < a + \varepsilon\}$. Since $G(z)$ is holomorphic everywhere in C^+ and C^- , it is holomorphic in $\Omega_\varepsilon \setminus \mathbb{R}$. In addition, by assumption (2), $G(z)$ is holomorphic at each point of $\Omega_\varepsilon \cap \mathbb{R}$. Therefore, it is holomorphic everywhere in Ω_ε and thus it is holomorphic everywhere in $\Omega_\varepsilon \cup \{z \mid |z| > a + \varepsilon/2\} = \{z \mid |z| > a - \varepsilon\}$. This contradicts the definition of a . Therefore, $a \leq L$ and $G(z)$ is holomorphic everywhere in the area $\Omega = \{z \mid |z| > L\}$, including the point at infinity.

It follows that the power series (2) converges everywhere in the area $\Omega = \{z \mid |z| > L\}$. Since this power series has real coefficients, we can conclude that $G(z)$ is real for $z \in \mathbb{R}$, $|z| > L$. Also, since $G(z)$ is holomorphic, and therefore continuous, in $|z| > L$, we can conclude that $\lim_{\varepsilon \downarrow 0} \text{Im } G(z + i\varepsilon) = 0$. Then the Stieltjes inversion formula implies $\mu([a, b]) = 0$ for each $[a, b] \subset \{x \in \mathbb{R} \mid |x| > L\}$ provided that $\mu(a) = 0$ and $\mu(b) = 0$. It remains to prove that this implies $\mu\{|x| > L\} = 0$.

For this purpose, note that the set of points $x \in \mathbb{R}$ for which $\mu(x) > 0$ is at most countable. Indeed, let S be the set of all x for which $\mu(x) > 0$. We can divide this set into a countable collection of disjoint subsets S_k , where k are all positive integers and $S_k = \{x \mid k^{-1} \geq \mu\{x\} > (k + 1)^{-1}\}$. Clearly, every S_k is either empty or a finite set. Otherwise, we could take an infinite countable sequence of $x_{i,k} \in S_k$ and would get (by countable additivity and monotonicity of μ) that $\mu(S_k) \geq \sum_i \mu(x_{i,k}) = +\infty$. By monotonicity of μ , we would further get $\mu(\mathbb{R}) = +\infty$, which would contradict the assumption that μ is a probability measure. Therefore, S is a countable union of finite sets S_k and hence countable.

From the countability of S , we conclude that the set of points x for which $\mu(x) = 0$ (i.e., S^c) is dense in the set $|x| > L$. Indeed, take an arbitrary nonempty interval (α, β) . Then $(\alpha, \beta) \cap S^c \neq \emptyset$ since, otherwise, $(\alpha, \beta) \subset S$ and therefore S would be uncountable. Hence, S^c is dense. Using the denseness of S^c , we can cover the set $\{|x| > L\}$ by a countable union of disjoint intervals $[a, b]$, where $\mu(a) = 0$ and $\mu(b) = 0$. For each of these intervals, $\mu([a, b]) = 0$ and therefore countable additivity implies that $\mu(\{|x| > L\}) = 0$. Consequently, μ is supported on a set that lies entirely in $[-L, L]$. \square

DEFINITION 4. The inverse of the G -series (2) (in the sense of formal series) always exists and is called the K -series,

$$G(K(z)) = K(G(z)) = z.$$

In case of a bounded self-adjoint random variable A , the G -series are convergent for $|z| \geq \|A\|$ and the limit coincides with the Cauchy transform $G(z)$. As a consequence, the K -series is convergent in a sufficiently small punctured neighborhood of 0. We will call the limit $K(z)$. This function has a pole of order 1 at 0.

It is sometimes useful to know how functions $G(z)$ and $K(z)$ behave under a rescaling of the random variable.

LEMMA 2. (i) $G_{\alpha A}(z) = \alpha^{-1}G_A(z/\alpha)$ and (ii) $K_{\alpha A}(u) = \alpha K_A(\alpha u)$.

The claim of the lemma follows directly from definitions.

The importance of K -functions is that they allow us to compute the distribution of the sum of free random variables.

PROPOSITION 3 (Voiculescu's addition formula). *Suppose that self-adjoint r.v.'s A and B are free. Let K_A , K_B and K_{A+B} be the K -series for variables A , B and $A + B$, respectively. Then*

$$K_{A+B}(u) = K_A(u) + K_B(u) - \frac{1}{u},$$

where the equality holds in the sense of formal power series.

The proof can be found in [19]. Using this property, we can compute the distribution of the sum of free r.v.'s as follows. Given two r.v.'s, A and B , compute their G -series. Invert them to obtain the K -series. Use Proposition 3 to compute K_{A+B} and invert it to obtain G_{A+B} . Use the Stieltjes inversion formula to compute the measure corresponding to this G -series. This is the probability measure corresponding to $A + B$.

4. Proof of Theorem 1. The key ideas of the proof are as follows.

(1) We know that the Cauchy transform of the sum S_n is the Cauchy transform of a bounded r.v. (since, by assumption, each $X_{n,i}$ is bounded). Consequently, the Cauchy transform of S_n is holomorphic in a certain circle around infinity (i.e., in the area $|z| > R$ for some $R > 0$). We want to estimate R and apply Lemma 1 to conclude that S_n is supported on $[-R, R]$.

(2) Since the K -function of S_n , call it $K_n(z)$, is the sum of the K -functions of $X_{n,i}$ and the latter are functional inverses of the Cauchy transforms of $X_{n,i}$, it is an exercise in complex analysis to prove that the K -function of S_n takes real values and is a one-to-one function on a sufficiently large real interval around zero. Therefore, it has a differentiable functional inverse defined on a sufficiently large real interval around infinity (i.e., on the set $I = (-\infty, -A] \cup [A, \infty)$ for some A which we can explicitly estimate). Moreover, with a little bit more effort, we can

show that this inverse function is well defined and holomorphic in an open complex neighborhood of I . This shows that Lemma 1 is applicable, and the estimate for A provides the desired estimate for the support of S_n .

We will begin by finding the radius of convergence of the Taylor series of $K_n(z)$. First, we need to prove some preliminary facts about Cauchy transforms of $X_{n,i}$.

Define $g_{n,i}(z) = G_{n,i}(z^{-1})$. Since the series $G_{n,i}(z)$ are convergent everywhere in $|z| > L_{n,i}$, the Taylor series for $g_{n,i}(z)$ converges everywhere in $|z| < L_{n,i}^{-1}$.

Assume that $R_{n,i}$ and $m_{n,i}$ are such that:

1. $R_{n,i} \geq L_{n,i}$;
2. $|G_{n,i}(z)| \geq m_{n,i} > 0$ everywhere on $|z| = R_{n,i}$;
3. $g_{n,i}(z)$ has only one zero in $|z| < R_{n,i}^{-1}$.

For example, we can take $R_{n,i} = 2L_{n,i}$ and $m_{n,i} = (4L_{n,i})^{-1}$. Indeed, for any z with $|z| = r > L_{n,i}$, we can estimate $G_{n,i}(z)$:

$$\begin{aligned} |G_{n,i}(z)| &\geq \frac{1}{r} - \left(\frac{a_{n,i}^2}{r^3} + \frac{|a_{n,i}^3|}{r^4} + \dots \right) \\ &\geq \frac{1}{r} - \left(\frac{L_{n,i}^2}{r^3} + \frac{L_{n,i}^3}{r^4} + \dots \right) \\ &= \frac{1}{r} - \frac{L_{n,i}^2}{r^2} \frac{1}{r - L_{n,i}}. \end{aligned}$$

In particular, taking $r = 2L_{n,i}$, we obtain the estimate:

$$|G_{n,i}(z)| \geq \frac{1}{4L_{n,i}},$$

valid for every i and everywhere on $|z| = 2L_{n,i}$.

It remains to show that $g_{n,i}(z)$ has only one zero in $|z| < (2L_{n,i})^{-1}$. This is indeed so because

$$g_{n,i}(z) = z(1 + a_{n,i}^{(2)}z^2 + a_{n,i}^{(3)}z^3 + \dots)$$

and we can estimate

$$|a_{n,i}^{(2)}z^2 + a_{n,i}^{(3)}z^3 + \dots| \leq L_{n,i}^2 \left(\frac{1}{2L_{n,i}} \right)^2 + L_{n,i}^3 \left(\frac{1}{2L_{n,i}} \right)^3 + \dots = \frac{1}{2}.$$

Therefore, Rouché’s theorem is applicable and $g_{n,i}$ has only one zero in $|z| < (2L_{n,i})^{-1}$.

DEFINITION 5. Let $R_n = \max_i \{R_{n,i}\}$, $m_n = \min_i \{m_{n,i}\}$ and $D_n = \sum_{i=1}^{k_n} R_{n,i}(m_{n,i})^{-2}$.

We are now able to investigate the region of convergence for the series $K_{n,i}(z)$. First, we prove a modification of Lagrange’s inversion formula.

LEMMA 3. *Suppose $w = G(z)$ (where G is not necessarily a Cauchy transform) is holomorphic in a neighborhood of $z_0 = \infty$ and has the expansion*

$$G(z) = \frac{1}{z} + \frac{a_1}{z^2} + \dots,$$

converging for all sufficiently large z . Define $g(z) = G(1/z)$. Then the inverse of $G(z)$ is well defined in a neighborhood of 0 and its Laurent series at 0 is given by the formula

$$z = G^{-1}(w) = \frac{1}{w} + a_1 - \sum_{n=1}^{\infty} \left[\frac{1}{2\pi i n} \oint_{\partial\gamma} \frac{dz}{z^2 g(z)^n} \right] w^n,$$

where γ is a sufficiently small disc around 0.

PROOF. Let γ be a closed disc around $z = 0$ in which $g(z)$ has only one zero. This disc exists because $g(z)$ is holomorphic in a neighborhood of 0 and has a nonzero derivative at 0. Let

$$r_w = \frac{1}{2} \inf_{z \in \partial\gamma} |g(z)|.$$

Then $r_w > 0$, by our assumption on γ . We can apply Rouché’s theorem and conclude that the equation $g(z) - w = 0$ has only one solution inside γ if $|w| \leq r_w$. Let us consider a w such that $|w| \leq r_w$. Inside γ , the function

$$\frac{g'(z)}{z(g(z) - w)}$$

has a pole at $z = 1/G^{-1}(w)$ with the residue $G^{-1}(w)$ and a pole at $z = 0$ with the residue $-1/w$. Consequently, we can write:

$$G^{-1}(w) = \frac{1}{2\pi i} \oint_{\partial\gamma} \frac{g'(z) dz}{z(g(z) - w)} + \frac{1}{w}.$$

The integral can be rewritten as follows:

$$\begin{aligned} \oint_{\partial\gamma} \frac{g'(z) dz}{z(g(z) - w)} &= \oint_{\partial\gamma} \frac{g'(z)}{zg(z)} \frac{1}{1 - w/g(z)} dz \\ &= \sum_{n=0}^{\infty} \oint_{\partial\gamma} \frac{g'(z) dz}{zg(z)^{n+1}} w^n. \end{aligned}$$

For $n = 0$, we calculate

$$\frac{1}{2\pi i} \oint_{\partial\gamma} \frac{g'(z) dz}{zg(z)} = a_1.$$

Indeed, the only pole of the integrand is at $z = 0$, of order two, and the corresponding residue can be computed from the series expansion for $g(z)$:

$$\begin{aligned} \operatorname{res}_{z=0} \frac{g'(z) dz}{zg(z)} &= \frac{d}{dz} \frac{z^2(1 + 2a_1z + \dots)}{z(z + a_1z^2 + \dots)} \Big|_{z=0} \\ &= \frac{d}{dz} \frac{1 + 2a_1z + \dots}{1 + a_1z + \dots} \Big|_{z=0} = a_1. \end{aligned}$$

For $n > 0$, we integrate by parts:

$$\frac{1}{2\pi i} \oint_{\partial\gamma} \frac{g'(z) dz}{zg(z)^{n+1}} = -\frac{1}{2\pi i} \frac{1}{n} \oint_{\partial\gamma} \frac{dz}{z^2 g(z)^n}. \quad \square$$

LEMMA 4. *The radius of convergence of the K -series for measure μ_n is at least m_n .*

The lemma essentially says that if r.v.'s $X_{n,1}, \dots, X_{n,k_n}$ are all bounded by L_n , then the K -series for $\sum_i X_{n,i}$ converges in the circle $|z| \leq 1/(4L_n)$.

PROOF. Let us apply Lemma 3 to $G_{n,i}(z)$ with γ having radius $(R_{n,i})^{-1}$. By Lemma 3, the coefficients in the series for the inverse of $G_{n,i}(z)$ are

$$b_{n,i}^{(k)} = \frac{1}{2\pi i k} \oint_{\partial\gamma} \frac{dz}{z^2 g_{n,i}(z)^k}$$

and we can estimate them as

$$|b_{n,i}^{(k)}| \leq \frac{R_{n,i}}{k} (m_{n,i})^{-k}.$$

This implies that the radius of convergence of the K -series for measure $\mu_{n,i}$ is $m_{n,i}$. Consequently, the radius of convergence of the K -series for measure μ_n is at least m_n . \square

We can now investigate the behavior of $K_n(z)$ and its derivative inside its circle of convergence.

LEMMA 5. *For every z in $|z| < m_n$, the following inequalities are valid:*

$$(3) \quad \left| K_n(z) - \frac{1}{z} - v_n z \right| \leq D_n |z|^2,$$

$$(4) \quad \left| K'_n(z) + \frac{1}{z^2} - v_n \right| \leq 2D_n |z|.$$

Note that D_n is approximately $k_n L_n^3$, so the meaning of the lemma is that the growth of $K_n - z^{-1} - v_n z$ around $z = 0$ is bounded by a constant that depends on the norm of the variables $X_{n,1}, \dots, X_{n,k_n}$.

PROOF. Consider the circle of radius $m_{n,i}/2$. We can estimate $K_{n,i}$ inside this circle:

$$\begin{aligned} \left| K_{n,i} - \frac{1}{z} - a_{n,i}^{(2)}z \right| &\leq \frac{R_{n,i}}{2}(m_{n,i})^{-2}|z|^2 + \frac{R_{n,i}}{3}(m_{n,i})^{-3}|z|^2 \frac{m_{n,i}}{2} \\ &\quad + \frac{R_{n,i}}{4}(m_{n,i})^{-3}|z|^2 \frac{m_{n,i}^2}{2^2} + \dots \\ &= R_{n,i}(m_{n,i})^{-2}|z|^2 \left(\frac{1}{2} + \frac{1}{3} \frac{1}{2} + \frac{1}{4} \frac{1}{2^2} + \dots \right) \\ &\leq R_{n,i}(m_{n,i})^{-2}|z|^2. \end{aligned}$$

Consequently, using Voiculescu’s addition formula, we can estimate

$$(5) \quad \left| K_n(z) - \frac{1}{z} - v_n z \right| \leq D_n |z|^2$$

and a similar argument leads to the estimate

$$(6) \quad \left| K'_n(z) + \frac{1}{z^2} - v_n \right| \leq 2D_n |z|. \quad \square$$

LEMMA 6. Suppose that (i) $m_n > 4/\sqrt{v_n}$ and (ii) $r_n \geq 4D_n/v_n^2$. Then there are no zeros of $K'_n(z)$ inside $|z| < 1/\sqrt{v_n} - r_n$.

In other words, $K_n(z)$ has no critical points in a circle which is sufficiently separated from $z = \pm 1/\sqrt{v_n}$.

PROOF OF LEMMA 6. If $r_n \geq v_n^{-1/2}$, then the set $|z| < 1/\sqrt{v_n} - r_n$ is empty and we are done. In the following, we assume that $r_n < v_n^{-1/2}$. On $|z| = v_n^{-1/2} - r_n$, we have $|z|^{-2} > v_n$. Also, $|z - v_n^{-1/2}||z + v_n^{-1/2}| > r_n v_n^{-1/2}$. This is easy to see by considering the two cases $\text{Re } z \geq 0$ and $\text{Re } z \leq 0$. In the first case, $|z - v_n^{-1/2}| \geq r_n$ and $|z + v_n^{-1/2}| > v_n^{-1/2}$. In the second case, $|z - v_n^{-1/2}| > v_n^{-1/2}$ and $|z + v_n^{-1/2}| \geq r_n$. Hence, in both cases, the product $|z - v_n^{-1/2}||z + v_n^{-1/2}| > r_n v_n^{-1/2}$.

Therefore,

$$\begin{aligned} |-z^{-2} + v_n| &= v_n |z|^{-2} |z - v_n^{-1/2}||z + v_n^{-1/2}| \\ &> v_n v_n r_n v_n^{-1/2} = r_n v_n^{3/2}. \end{aligned}$$

Since $r_n + v_n^{-1/2} < 2v_n^{-1/2}$, assumption (i) implies that $r_n + v_n^{-1/2} < m_n/2$. Hence, the circle $|z| = v_n^{-1/2} + r_n$ lies entirely in the area where formula (4) applies to $K'_n(z)$. Consequently, using (4), we can estimate

$$|K'_n(z) - (-z^{-2} + v_n)| \leq 4D_n v_n^{-1/2},$$

where we used the fact that $|z| \leq 2v_n^{-1/2}$. By assumption (ii), $r_n \geq 4D_n v_n^{-2}$, therefore $v_n^{3/2} r_n \geq 4D_n v_n^{-1/2}$ and Rouché's theorem is applicable to the pair of $K'_n(z)$ and $-z^{-2} + v_n$. Both $K'_n(z)$ and $-z^{-2} + v_n$ have only one pole, of order two, in $|z| \leq v^{-1/2} - r_n$ and the function $-z^{-2} + v_n$ has no zeros inside $|z| \leq v^{-1/2} - r_n$. Therefore, Rouché's theorem implies that there are no zeros of $K'_n(z)$ inside $|z| \leq v^{-1/2} - r_n$. (Rouché's theorem is often formulated only for holomorphic functions, but as a consequence of the argument principle (see, e.g., Theorems II.2.3 and II.2.4 in [12]), it can be easily reformulated for meromorphic functions. In this form, it claims that a meromorphic function, $f(z)$, has the same difference between the number of zeros and number of poles inside a curve γ as another meromorphic function, $g(z)$, provided that $|f(z)| > |g(z) - f(z)|$. For this formulation see, e.g., [9], Theorem 9.2.3.) \square

CONDITION 2. Assume, in the following, that $r_n = 4D_n/v_n^2$.

We now use our knowledge about the location of critical points of $K_n(z)$ to investigate how it behaves on the real interval around zero.

LEMMA 7. Suppose that $m_n > 4/\sqrt{v_n}$ and $D_n/v_n^{3/2} \leq 1/8$. Then $K_n(z)$ maps the set $[-1/\sqrt{v_n} + r_n, 0) \cup (0, 1/\sqrt{v_n} - r_n]$ in a one-to-one fashion onto a set that contains the union of two intervals $(-\infty, -2\sqrt{v_n} - cD_n/v_n) \cup (2\sqrt{v_n} + cD_n/v_n, \infty)$, where c is a constant that does not depend on n .

REMARK. For example, $c = 5$ will work.

PROOF OF LEMMA 7. The assumption that $m_n > 4/\sqrt{v_n}$ ensures that the power series for $K_n(z)$ converges in $|z| \leq 4/\sqrt{v_n}$, $z \neq 0$. Note that $K_n(z)$ is real-valued on the set $I = [-1/\sqrt{v_n} + r_n, 0) \cup (0, 1/\sqrt{v_n} - r_n]$ because this set belongs to the area where the series for $K_n(z)$ converges and the coefficients of this series are real. Moreover, by Lemma 6, there are no critical points of $K_n(z)$ on I [i.e., for every $z \in I$, $K'_n(z) \neq 0$], therefore $K_n(z)$ must be strictly monotonic on subintervals $[-1/\sqrt{v_n} + r_n, 0)$ and $(0, 1/\sqrt{v_n} - r_n]$. Consequently, $K_n(I) = (-\infty, K_n(-1/\sqrt{v_n} + r_n)] \cup [K_n(1/\sqrt{v_n} - r_n), \infty)$. We claim that $K_n(1/\sqrt{v_n} - r_n) \leq 2\sqrt{v_n} + 5D_n/v_n$ and $K_n(-1/\sqrt{v_n} + r_n) \geq -2\sqrt{v_n} - 5D_n/v_n$.

Indeed, if we write

$$K_n(z) = \frac{1}{z} + v_n z + h(z),$$

then

$$K_n\left(\frac{1}{\sqrt{v_n}} - r_n\right) = \sqrt{v_n} \frac{1}{1 - r_n \sqrt{v_n}} + \sqrt{v_n}(1 - r_n \sqrt{v_n}) + h\left(\frac{1}{\sqrt{v_n}} - r_n\right).$$

According to our assumption, $r_n\sqrt{v_n} = 4D_n/v_n^{3/2} < 1/2$. Therefore, we can estimate

$$\frac{1}{1 - r_n\sqrt{v_n}} \leq 1 + 2r_n\sqrt{v_n}$$

and

$$K_n\left(\frac{1}{\sqrt{v_n}} - r_n\right) \leq 2\sqrt{v_n} + r_nv_n + \left|h\left(\frac{1}{\sqrt{v_n}} - r_n\right)\right|.$$

We can estimate the last term using Lemma 5 as

$$h\left(\frac{1}{\sqrt{v_n}} - r_n\right) \leq D_n\left|\frac{1}{\sqrt{v_n}}\right|^2 = D_n/v_n.$$

Combining all of this and substituting $r_n = 4D_n/v_n^2$, we get

$$K_n\left(\frac{1}{\sqrt{v_n}} - r_n\right) \leq 2\sqrt{v_n} + 5D_n/v_n.$$

Similarly, we can derive that

$$K_n\left(-\frac{1}{\sqrt{v_n}} + r_n\right) \geq -2\sqrt{v_n} - 5D_n/v_n. \quad \square$$

From the previous lemma, we can conclude that $K_n(z)$ has a differentiable inverse defined on $(-\infty, -2\sqrt{v_n} - cD_n/v_n) \cup (2\sqrt{v_n} + cD_n/v_n, \infty)$. We can extend this conclusion to an open complex neighborhood of this interval. This is achieved in the next two lemmas.

LEMMA 8. *As in the previous lemma, suppose that $m_n > 4/\sqrt{v_n}$ and $D_n/v_n^{3/2} \leq 1/8$. Let z be an arbitrary point of the interval $[-1/\sqrt{v_n} + r_n, 1/\sqrt{v_n} - r_n]$. Then we can find a neighborhood U_z of z and a neighborhood W_w of $w = K_n(z)$ such that K_n is a one-to-one map of U_z onto W_w and the inverse map K_n^{-1} is holomorphic everywhere in W_w .*

PROOF. Since the power series for $K_n(z) - z^{-1}$ converges in $|z| \leq 4/\sqrt{v_n}$, the function $K_n(z)$ is holomorphic in $|z| \leq 4/\sqrt{v_n}$, $z \neq 0$. In addition, by Lemma 6, $z \in [-1/\sqrt{v_n} + r_n, 1/\sqrt{v_n} - r_n]$ is not a critical point of $K_n(z)$. Therefore, for $z \neq 0$, the conclusion of the lemma follows from Theorems II.3.1 and II.3.2 in [12]. For $z = 0$, the argument is parallel to the argument in Markushevich, except for a different choice of local coordinates. Indeed, $f(z) = 1/K_n(z)$ is holomorphic at $z = 0$, it maps $z = 0$ to $w = 0$ and $f'(z) = 1 \neq 0$ at $z = 0$. Therefore, Theorems II.3.1 and II.3.2 in [12] are applicable to $f(z)$ and it has a well-defined holomorphic inverse in a neighborhood of $w = 0$. This implies that $K_n(z)$ has a well-defined holomorphic inverse in a neighborhood of ∞ , given by the formula $K_n^{-1}(z) = f^{-1}(1/z)$. \square

LEMMA 9. *Local inverse $K_n^{-1}(z)$ defined in the previous lemma is a restriction of a function $G_n(z)$ which is defined and holomorphic everywhere in a neighborhood of $I = \{\infty\} \cup (-\infty, -2v_n^{1/2} - cD_n/v_n] \cup [2v_n^{1/2} + cD_n/v_n, \infty)$. The function $G_n(z)$ is the inverse of $K_n(z)$ in this neighborhood.*

PROOF. By Lemma 7, for every point $w \in I$, we can find a unique $z \in [-1/\sqrt{v_n} + r_n, 1/\sqrt{v_n} - r_n]$ such that $K_n(z) = w$. Let U_z and W_w be the neighborhoods defined in the previous lemma. Also, let us write (K_n^{-1}, W_w) to denote the local inverses defined in the previous lemma together with their areas of definition. Our task is to prove that these local inverses can be joined to form an analytic function K_n^{-1} , well defined everywhere in a neighborhood of I . We will do this in several steps.

First, an examination of the proof of the previous lemma and Theorem II.3.1 in [12] shows that we can take each U_z in the form of a disc. Let $\tilde{U}_z = U_z/3$, that is, define \tilde{U}_z as a disc that has the same center, but radius one third that of U_z . Define \tilde{W}_w as $K_n(\tilde{U}_z)$. These new sets are more convenient because of the property that if $\tilde{U}_{z_1} \cap \tilde{U}_{z_2} \neq \emptyset$, then either $\tilde{U}_{z_1} \cup \tilde{U}_{z_2} \subset U_{z_1}$ or $\tilde{U}_{z_1} \cup \tilde{U}_{z_2} \subset U_{z_2}$. In particular, this means that if $\tilde{U}_{z_1} \cap \tilde{U}_{z_2} \neq \emptyset$, then $K_n(z)$ is a one-to-one map of $\tilde{U}_{z_1} \cup \tilde{U}_{z_2}$ onto $\tilde{W}_{w_1} \cup \tilde{W}_{w_2}$. This is convenient because K_n is one-to-one not only on a particular neighborhood \tilde{U}_{z_1} , but also on the union of every two intersecting neighborhoods \tilde{U}_{z_1} and \tilde{U}_{z_2} . Let us call this the *extended invertibility* property.

Next, define an even smaller $\tilde{\tilde{U}}_z$ with the following properties: (1) $\tilde{\tilde{U}}_z \subset \tilde{U}_z$; (2) $\tilde{\tilde{W}}_w =: K_n(\tilde{\tilde{U}}_z)$ is either an open disc for $z \neq 0$ or the set $|w| > R$ for $z = 0$; and (3) $0 \notin \tilde{\tilde{W}}_w$. This is easy to achieve by taking an appropriate open subset of \tilde{W}_w as $\tilde{\tilde{W}}_w$ and applying K_n^{-1} . Note that the property of the previous paragraph (i.e., extended invertibility) remains valid for the new sets $\tilde{\tilde{U}}_z$.

Discs $\tilde{\tilde{W}}_w$ form an open cover of I and the corresponding sets $\tilde{\tilde{U}}_z$ form an open cover for $K_n^{-1}(I)$, which is a closed interval contained in $[-1/\sqrt{v_n} + r_n, 1/\sqrt{v_n} - r_n]$. Let $U_i, i = 0, \dots, N$, be a finite cover of $K_n^{-1}(I)$ selected from $\{\tilde{\tilde{U}}_z\}$. [Recall that K_n^{-1} is well defined on the interval I by Lemma 7, hence $K_n^{-1}(I)$ is well defined.] We can find a finite cover due to the compactness of $K_n^{-1}(I)$. Further, let $W_i =: K_n(U_i)$ be the corresponding cover of I , selected from $\{\tilde{\tilde{W}}_z\}$. For convenience, let W_0 denote the set $\tilde{\tilde{W}}_w$ for $w = \infty$. Finally, let $R = \bigcup_{i=0}^N U_i$ and $S = \bigcup_{i=0}^N W_i$. Sets R and S are illustrated in Figure 1.

Clearly, S is open. We aim to prove that S is simply connected in the extended complex plane $\mathbb{C} \cup \{\infty\}$. For this purpose, let us define the deformation retraction F_t of the set S as follows: (1) if $z \in W_0$, then $z \rightarrow z$; (2) if $z \notin W_0$, then $z \rightarrow \text{Re } z + (1 - t) \text{Im } z$. Here, parameter t changes from 0 to 1. (For the definition and properties of deformation retractions, see, e.g., [8]; the definition is on page 2 and the main property is in Proposition 1.17.) This retraction reduces S to a homotopically equivalent set S' that consists of W_0 and two intervals of the real axis that

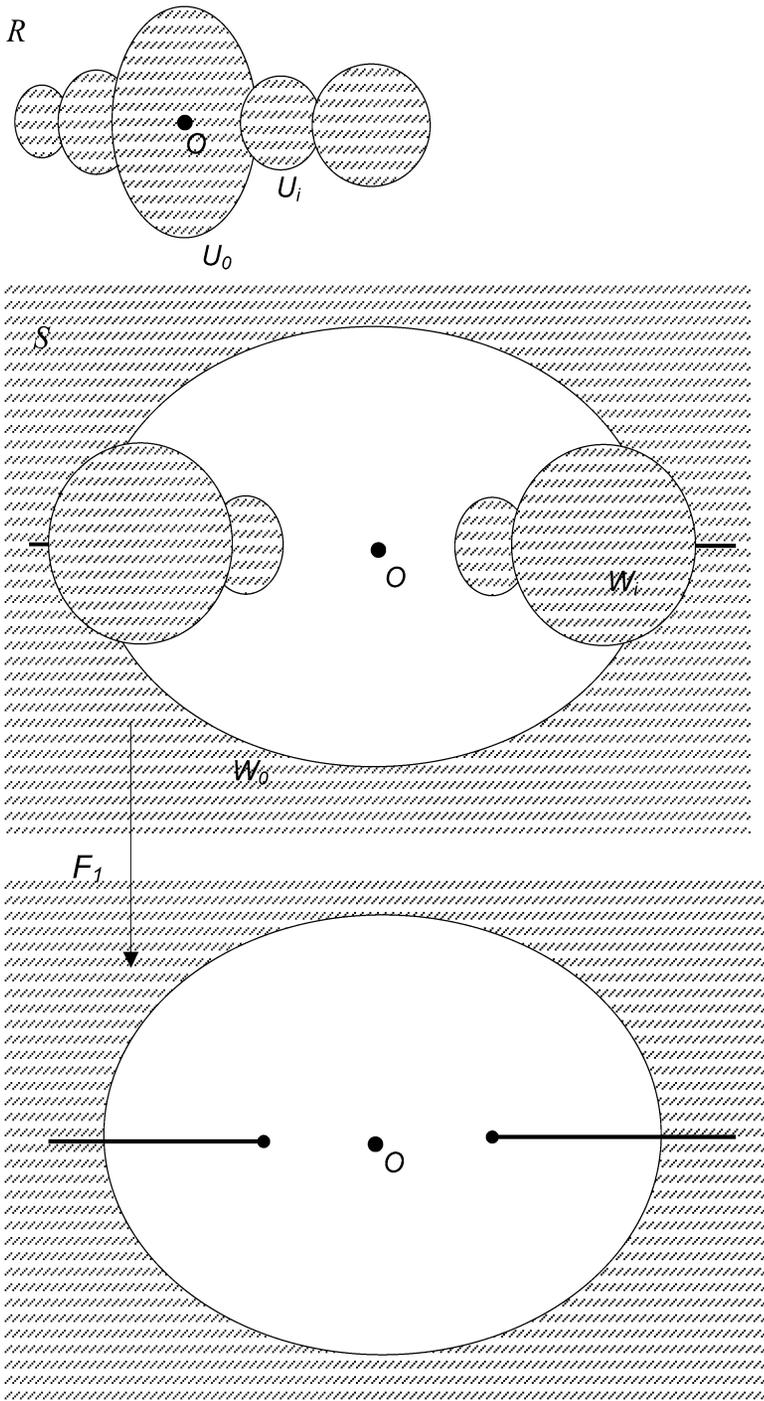


FIG. 1.

do not include 0. We can then use another deformation retraction F_2 that sends z to $(1 - t)^{-1}z$. This retraction reduces S' to $S'' = \{\infty\}$, which is evidently simply connected.

We know that there is a holomorphic inverse $K_n^{-1}(z)$ defined on each of W_i . Starting from one of these domains, say W_0 , we can analytically continue $K_n^{-1}(z)$ to every other W_i . Indeed, take points $z_0 \in U_0$ and $z_i \in U_i$ and connect them by a path that lies entirely in $R = \bigcup_{i=0}^N U_i$. This path corresponds to a chain $\{U_{k_s}\}$, $s = 1, \dots, n$, that connects U_0 and U_i . That is, $U_{k_1} = U_0$, $U_{k_n} = U_i$ and $U_{k_j} \cap U_{k_{j+1}} \neq \emptyset$. The corresponding $W_{k_s} = K_n(U_{k_s})$ form a chain that connects W_0 and W_j , that is, $W_{k_1} = W_0$, $W_{k_n} = W_i$ and $W_{k_j} \cap W_{k_{j+1}} \neq \emptyset$. By our construction, this chain of sets W_{k_s} has the property that $K_n^{-1}(W_{k_j}) \cap K_n^{-1}(W_{k_{j+1}}) = U_{k_j} \cap U_{k_{j+1}} \neq \emptyset$.

Consider two adjacent sets, W_{k_j} and $W_{k_{j+1}}$, in this chain. Then the corresponding local inverses (K_n^{-1}, W_{k_j}) and $(K_n^{-1}, W_{k_{j+1}})$, which were defined in the previous lemma, coincide on an open nonempty set. Indeed, $K_n(U_{k_j} \cap U_{k_{j+1}}) \subset K_n(U_{k_j}) \cap K_n(U_{k_{j+1}}) = W_{k_j} \cap W_{k_{j+1}}$, therefore the functions (K_n^{-1}, W_{k_j}) and $(K_n^{-1}, W_{k_{j+1}})$ are both well defined on $K_n(U_{k_j} \cap U_{k_{j+1}})$. Moreover, they must coincide on $K_n(U_{k_j} \cap U_{k_{j+1}})$. Indeed, by construction, $U_{k_j} \cap U_{k_{j+1}} \neq \emptyset$ and therefore, by the extended invertibility property, K_n is one-to-one on $U_{k_j} \cup U_{k_{j+1}}$. Hence, there cannot exist two different z and z' in $U_{k_j} \cup U_{k_{j+1}}$ that would map to the same point in $K_n(U_{k_j} \cap U_{k_{j+1}})$. Hence, (K_n^{-1}, W_{k_j}) and $(K_n^{-1}, W_{k_{j+1}})$ must coincide on $K_n(U_{k_j} \cap U_{k_{j+1}})$, which is open and nonempty.

Using the property that if two analytical functions coincide on an open set, then each of them is an analytic continuation of the other, we conclude that the local inverse (K_n^{-1}, W_{k_j}) can be analytically continued to $W_{k_{j+1}}$, where it coincides with the local inverse $(K_n^{-1}, W_{k_{j+1}})$. Therefore, at least one analytic continuation of (W_0, K_n^{-1}) is well defined everywhere on S and has the property that when restricted to each of W_j , it coincides with a local inverse of $K_n(z)$ defined in the previous lemma. Since S is simply connected, the analytic continuation is unique, that is, it does not depend on the choice of the chain of the neighborhoods that connect W_0 and W_j .

Let us denote the function resulting from this analytic continuation as $G_n(z)$. By construction, it is unambiguously defined for every W_j and the restrictions of $G_n(z)$ to W_j coincide with (K_n^{-1}, W_j) . Therefore, $G_n(z)$ satisfies the relations $K_n(G_n(z)) = z$ and $G_n(K_n(z)) = z$ everywhere on $R = \bigcup_{i=0}^N U_i$ and on $S = \bigcup_{i=0}^N W_i$. Since S is an open neighborhood of I , every claim of the lemma is proved. \square

LEMMA 10. *The function $G_n(z)$ constructed in the previous lemma is the Cauchy transform of S_n .*

By construction, $G_n^{-1}(z)$ is the inverse of $K_n(z)$ in a neighborhood of $\{\infty\} \cup (-\infty, -2v_n^{1/2} - cD_n/v_n) \cup (2v_n^{1/2} + cD_n/v_n, \infty)$. In particular, it is the inverse of $K_n(z)$ in a neighborhood of infinity. Therefore, in this neighborhood, it has the same power expansion as the Cauchy transform of S_n . Therefore, it coincides with the Cauchy transform of S_n in this neighborhood. Next, we apply the principle that if two analytical functions coincide in an open domain, then they coincide at every point where they can be continued analytically.

It remains to apply Lemma 1 in order to obtain the following theorem.

THEOREM 3. *Suppose that (i) $\liminf m_n \sqrt{v_n} > 4$ and (ii) $\limsup_{n \rightarrow \infty} D_n/v_n^{3/2} \leq 1/8$. Then for all sufficiently large n , the support of μ_n belongs to*

$$I = (-2\sqrt{v_n} - cD_n/v_n, 2\sqrt{v_n} + cD_n/v_n),$$

where $c > 0$ is an absolute constant (e.g., $c = 5$).

PROOF OF THEOREM 3. Let us collect the facts that we know about the function $G_n(z)$ defined in Lemma 9. First, by Lemma 10, it is the Cauchy transform of a bounded random variable, S_n . Second, by Lemma 9, it is holomorphic at all $z \in \mathbb{R}$ such that $|z| > 2v_n^{1/2} + cD_n/v_n$. Using Lemma 1, we conclude that the distribution of S_n is supported on the interval $[-2v_n^{1/2} - cD_n/v_n, 2v_n^{1/2} + cD_n/v_n]$. \square

If we take $R_{n,i} = 2L_{n,i}$ and $m_{n,i} = (4L_{n,i})^{-1}$, then assumption (i) is equivalent to

$$\liminf_{n \rightarrow \infty} \min_i \frac{\sqrt{v_n}}{4L_{n,i}} > 4,$$

which is equivalent to

$$\limsup_{n \rightarrow \infty} \frac{L_n}{\sqrt{v_n}} < 16.$$

From (ii), we obtain

$$1/8 \geq \limsup_{n \rightarrow \infty} \frac{\sum_{i=1}^{k_n} R_{n,i} (m_{n,i})^{-2}}{v_n^{3/2}} = \limsup_{n \rightarrow \infty} \frac{32 \sum_{i=1}^{k_n} L_{n,i}^3}{v_n^{3/2}},$$

which is equivalent to

$$\limsup_{n \rightarrow \infty} \frac{T_n}{v_n^{3/2}} \leq 1/256.$$

Finally, note that the condition $\limsup_{n \rightarrow \infty} T_n/v_n^{3/2} \leq 2^{-12}$ implies that $\limsup_{n \rightarrow \infty} L_n/\sqrt{v_n} < 16$. Therefore, Theorem 1 is a consequence of Theorem 3.

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