LaTeXsubmissions are mandatory. The template for this problem can be found on the Piazza resource page for this course.

Problem 1
True or false

1. If vectors v_1, v_2 , . . . , v_n generate (span) the vector space V, then every vector in V can be written as a linear combination of vector v_1, v_2, . . . , v_n in only one way;
2. If V is a vector space having dimension n, then V has exactly one subspace of dimension 0 and exactly one subspace of dimension n.
3. If columns of $m \times n$ matrix A span $\mathbb{R}^m$, then $Ax = b$ is consistent (i.e., has a solution) for every b.
4. If A is a 6×4 matrix with 4 pivotal rows, then $Ax = 0$ has infinitely many solutions.
5. It is possible that $n \times n$ matrix A has a pivot in every row and the system $Ax = b$ is inconsistent, where a and b are vectors in $\mathbb{R}^n$.
6. If the homogeneous system $Ax = 0$ corresponding to a given system of a linear equations $Ax = b$ has a solution, then the given system has a solution;
7. If the coefficient matrix of a homogeneous system of n linear equations in n unknowns is invertible, then the system has no non-zero solution;
8. The solution set of any system of m equations in n unknowns is a subspace in $\mathbb{R}^n$.

Solution:

Problem 2
Find a 2×3 system (2 equations with 3 unknowns) $Ax = b$, such that its general solution has a form:

$$x = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad s \in \mathbb{R}$$

Solution:

Problem 3
Find all vectors that are perpendicular to $[1, 4, 4, 1]^t$ and $[2, 9, 8, 2]^t$.

Solution:

Problem 4
Apply the Gram-Schmidt process to vectors $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, $\boldsymbol{a}_3$ and find orthonormal vectors $\boldsymbol{q}_1$, $\boldsymbol{q}_2$, $\boldsymbol{q}_3$.

$$\boldsymbol{a}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \boldsymbol{a}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \boldsymbol{a}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix},$$

Write matrix A as QR decomposition $A = QR$.

Solution:

Problem 5
Let A be a real symmetric matrix (i.e. $A^t = A$). An *eigenvector* of matrix A is a non-zero vector x such that $Ax = \lambda x$ for some number λ which is called the *eigenvalue* corresponding to the eigenvector x.

Prove that if x_1 and x_2 are eigenvectors corresponding to distinct real eigenvalues λ_1 and λ_2, respectively, then x_1 and x_2 are orthogonal. (Distinct means $\lambda_1 \neq \lambda_2$.)

Solution:

Problem 6
Let u and v are two column vectors in $\mathbb{R}^n$. The matrix $A = I + uv^t$ is known as a *rank-one perturbation of the identity*. Check that A has the inverse $A^{-1} = I + \alpha uv^t$ for some scalar α provided that A is non-singular and give an expression for α. For what u and v is A singular? If it is singular, what is nullspace(A)?

Solution:

Problem 7
Prove or disprove: If the columns of a square ($n \times n$) matrix A are linearly independent, so are the columns of $A^2 = AA$.

Solution: