Math 447 - March 12, 2013 - Test 2 Solutions

Name:______________________________

Read these instructions carefully: The points assigned are *not* meant to be a guide to the difficulty of the problems. If the question is multiple choice, there is a penalty for wrong answers, so that your expected score from guessing at random is zero. No partial credit is possible on multiple-choice and other no-work-required questions.

1. (18 points) (Monty Hall, Episode 2) Having won \$100 by selecting the queen of hearts ($Q\heartsuit$) on the previous episode of our game show, you have been invited back. Today there is a new twist: you are shown six cards, all face-down, of which one is the $Q\heartsuit$, three are black spot cards, and the other two are the black queens ($Q\spadesuit, Q\clubsuit$). The objective is to select the $Q\heartsuit$, which wins \$100. However, selecting either black queen carries a penalty of \$50. (You still have the \$100 you won in the previous episode, so you can pay.)

As usual, the host lets you select a face-down card, and then shows you a black spot card, which he can always do, because he knows where all the cards are. He offers then you the opportunity to switch your choice to one of the four remaining face-down cards.

1.1 (6 points) Suppose you decide to switch, and that you select one of the remaining four face-down cards at random. Let W be the event that your new selection is the (winning) $Q\heartsuit$. Which of the following numbers is closest to $P(W)$?

(a) 0.08
(b) 0.12
(c) 0.25
(d) 0.33

Solution. (c). Let H be the event that the inital selection is the $Q\heartsuit$, let B be the event that the initial selection is a black spot card, and let C be the event that the initial selection is the $Q\clubsuit$ or $Q\spadesuit$. Use the law of total probability to get

$$P(W) = P(W \mid H)P(H) + P(W \mid B)P(B) + P(W \mid C)P(C),$$

and then note that the conditional probabilities are 0, $\frac{1}{4}$, and $\frac{1}{4}$. Thus

$$P(W) = 0 \cdot \frac{1}{6} + \frac{1}{4} \cdot \frac{3}{6} + \frac{1}{4} \cdot \frac{2}{6} = \frac{5}{24} \approx 0.2083.$$

1.2 (6 points) Again suppose (as in 1.1) you decide to switch, and that you select one of the remaining four face-down cards at random. Let L be the event that your new selection is one of the (losing) black queens. Which of the following numbers is closest to $P(L)$?

(a) 0.50
(b) 0.40
(c) 0.33
(d) 0.25

Solution. (b). Let H be the event that the inital selection is the $Q\heartsuit$, let B be the event that the initial selection is a black spot card, and let C be the event that the initial selection is the $Q\clubsuit$ or $Q\spadesuit$. Use the law of total probability to get

$$P(L) = P(L \mid H)P(H) + P(L \mid B)P(B) + P(L \mid C)P(C),$$

and then note that the conditional probabilities are $\frac{1}{2}$, $\frac{1}{2}$, and $\frac{1}{4}$. Thus

$$P(L) = \frac{1}{2} \cdot \frac{1}{6} + \frac{1}{2} \cdot \frac{3}{6} + \frac{1}{4} \cdot \frac{2}{6} = \frac{5}{12} \approx 0.4167.$$

1.3 (6 points) Which of the following statements is most accurate?

(a) The decision to switch cards increases the probability of getting a losing card, and decreases the expected value from playing the game.

(b) The decision to switch cards increases the probability of getting a winning card, and increases the expected value from playing the game.

(c) The decision to switch cards decreases the probability of getting a losing card, but does not change the expected value from playing the game.

(d) The decision to switch cards increases the probability of getting a winning card, and increases the probability of getting a losing card.

Solution. (d): whether or not you decide to switch, your expectation is zero. The decision to switch increases the probability of winning and of losing, since $\frac{5}{24} > \frac{1}{6}$ and $\frac{5}{12} > \frac{2}{6}$.

2. (12 points) (Monty Hall, Episode 3) Having won another \$100 in Episode 2 by selecting the $Q\heartsuit$ again, you are invited back for Episode 3. Once again you are shown six cards, all face-down, of which one is the $Q\heartsuit$, three are black spot cards, and the other two are $Q\spadesuit, Q\clubsuit$. The prize for selecting the $Q\heartsuit$ is again \$100, and the penalty for a black queen (as before) \$50.

Once again you select a card at random from one of the six face-down cards. And now comes the twist of Episode 3: your host offers, for a fee, to show you a black queen instead of showing a black spot card. (Remember, the host can always do this because he knows where all the cards are.) After you pay the fee you may switch your selection to one of the four remaining face-down cards.

2.1 (6 points) Suppose you decide to pay the fee, see a black queen, switch, and that you select one of the remaining four face-down cards at random. Let L be the event that your new selection is the remaining (losing) black queen. Which of the following numbers is closest to $P(L)$?

(a) 0.18

(b) 0.12

(c) 0.06

(d) 0.24

Solution. (a). Let H be the event that the inital selection is the $Q\heartsuit$, let B be the event that the initial selection is a black spot card, and let C be the event that the initial selection is the $Q\clubsuit$ or $Q\spadesuit$. Use the law of total probability to get

$$P(L) = P(L \mid H)P(H) + P(L \mid B)P(B) + P(L \mid C)P(C),$$

and then note that the conditional probabilities are $\frac{1}{4}$, $\frac{1}{4}$, and $\frac{0}{4}$. Thus

$$P(L) = \frac{1}{4} \cdot \frac{1}{6} + \frac{1}{4} \cdot \frac{3}{6} + \frac{0}{4} \cdot \frac{2}{6} = \frac{4}{24} \approx 0.1667.$$

2.2 (6 points) Which of the following statements about the fee to see a black queen is most accurate?

(a) It would increase your expectation to pay a \$15 fee to see a black queen, but paying \$20 would decrease your expectation.

(b) Seeing the black queen instead of a black spot card decreases your expectation, so any fee paid is too much.

(c) Paying a fee of either \$15 or \$20 will increase your expectation, and you should be willing to pay either.

(d) Paying a fee of either \$15 or \$20 will decrease your expectation, so you should be willing to pay neither.

Solution. (d). With notation as in 1.1, use the law of total probability to get

$$P(W) = P(W \mid H)P(H) + P(W \mid B)P(B) + P(W \mid C)P(C),$$

and then note that the conditional probabilities are 0, $\frac{1}{4}$, and $\frac{1}{4}$. Thus

$$P(W) = 0 \cdot \frac{1}{6} + \frac{1}{4} \cdot \frac{3}{6} + \frac{1}{4} \cdot \frac{2}{6} = \frac{5}{24} \approx 0.2083.$$

Thus, switching gives an expected win of

$$\$(100 \cdot \frac{5}{24} - 50 \cdot \frac{4}{24}) = 12.50,$$

while the expectation of sticking with the original choice is zero.

3. (20 points) As items come to the end of a production line, an inspector chooses which items are to go through a complete inspection. Ten percent of all items produced are defective. Sixty percent of all defective items go through a complete inspection, and 20% of all good items go through a complete inspection. Given that an item is completely inspected, what is the probability it is defective?

a. (4 points) Establish notation: name the relevant events.

Solution. Let D be the event than an item is defective and let C be the event that an item goes through a complete inspection.

b. (4 points) What does the statement of the problem tell you about the probabilities of the events named in (a)?

Solution. $P(D) = 0.10$, $P(C \mid D) = 0.60$, and $P(C \mid \bar{D}) = 0.20$.

c. (4 points) Give an equation relating the probabilities in (b) to the desired answer.

Solution.

$$P(D \mid C) = \frac{P(C \mid D)P(D)}{P(C \mid D)P(D) + P(C \mid \bar{D})P(\bar{D})}$$

d. (8 points) Solve the problem: given that an item is completely inspected, what is the probability it is defective?

Solution. Plug the numbers from (b) into the formula from (c).

$$P(D \mid C) = \frac{(0.60)(0.10)}{(0.60)(0.10) + (0.20)(0.90)} = 0.25$$

4. (12 points) Let Y be a random variable with $p(y)$ given in the table below.

y	1	2	3	4
$p(y)$	0.4	0.3	0.2	0.1

a. (4 points) Find $E[Y]$

Solution. 2.0

b. (4 points) Find $E[Y^2]$

Solution. 5.0

c. (4 points) Find $V[Y]$

Solution. $5 - 2^2 = 1$.

5. (9 points) (A best-of-9 series) Two teams A and B play a series of games until one team wins 5 games. (Tied games are not possible.) The results of the games are independent and the probability of A winning each game is 0.7. The probability that the series lasts exactly 6 games is closest to

(a) 0.16

(b) 0.24

(c) 0.34

(d) 0.61

Solution. (b) For A to win the series in exactly 6 games, the first 5 games must be split 4–1, and then A must win the seventh game. Using our knowledge of the binomial random variable (Y, the number of games A wins in the first 5) we have

$$P(Y = 4) = \binom{5}{1}(0.7)^4(0.3)^1$$

Thus the probability that A wins the series is 6 games is $(0.7) \cdot \binom{5}{1}(0.7)^4(0.3)^1$. There is a similar formula for the probability that B wins the series in 6 games, and these two events are mutually exclusive, so the probability that the series ends in 6 games is

$$\binom{5}{1}(0.7)^5(0.3)^1 + \binom{5}{1}(0.3)^5(0.7)^1 \approx 0.245.$$

6. (20 points) Two people took turns tossing a fair die until one of them tossed a 6. Person A tossed first, B second, A third, and so on. Given that person B threw the first 6, what is the probability that B obtained the first 6 on her third toss (that is, on the sixth toss overall)?

a. (3 points) Let B be the event that B throws the first 6. Let T be the event that B throws the first 6 on her third toss. In terms of B and T, what conditional probability does the problem ask for?

Solution. $P(T \mid B)$.

b. (3 points) Give the definition of the conditional probability referred to in (a).

Solution.

$$P(T \mid B) = \frac{P(T \cap B)}{P(B)}$$

c. (4 points) Find $P(B \cap T)$.

Solution. Note that $B \cap T = T$, which is the event that the first six occurs on the sixth toss. Using our knowledge of the geometric random variable, this is $(\frac{5}{6})^5(\frac{1}{6}) \approx 0.067$.

d. (10 points) Solve the problem, i.e. find the conditional probability referred to in (a).

Solution. We must find

$$P(B) = \left((\frac{5}{6})^1(\frac{1}{6}) + (\frac{5}{6})^3(\frac{1}{6}) + (\frac{5}{6})^5(\frac{1}{6}) + \cdots\right) = \frac{5}{36}\sum_{y=0}^{\infty}(\frac{5}{6})^{2y} = \frac{5}{36}\left(\frac{1}{1-(\frac{5}{6})^2}\right) = \frac{5}{11}.$$

Now

$$P(T \mid B) = \frac{0.067}{\frac{5}{11}} \approx 0.1474.$$

7. (16 points) In southern California, a growing number of individuals pursuing teaching credentials are choosing paid internships over traditional student teaching programs. A group of eight candidates for three local teaching positions consisted of five who had enrolled in paid internships and three who enrolled in traditional student teaching programs. All eight candidates appear to be equally qualified, so three are randomly selected to fill the open positions. Let Y be the number of internship trained candidates who are hired.

a. (4 points) The distribution of Y is

(a) binomial
(b) negative binomial
(c) Poisson
(d) geometric
(e) hypergeometric

Solution. (e): Y is not binomial because we are sampling without replacement from a small population.

b. (4 points) Find the probability that exactly two internship-trained candidates are hired.

Solution. Use the probability function for the hypergeometric distribution, with $N = 8$, $n = 3$, $r = 5$.

$$P(Y = y) = \frac{\binom{r}{y}\binom{N-r}{n-y}}{\binom{N}{n}} = \frac{\binom{5}{2}\binom{3}{1}}{\binom{8}{3}} \approx 0.5357$$

c. (4 points) Find $E[Y]$.

Solution. $E[Y] = nr/N = 1.875$.

d. (4 points) Find $V[Y]$.

Solution.

$$V[Y] = n\frac{r}{N}\frac{N-r}{N}\frac{N-n}{N-1} \approx 0.5022$$

8. (20 points) Let Y be a Poisson random variable with parameter λ.

a. (4 points) Give the probability function $p(y) = P(Y = y)$.

Solution. $p(y) = e^{-\lambda}\frac{\lambda^y}{y!}$.

b. (4 points) Give the definition of the moment generating function $m(t)$ of Y.

Solution. $m(t) = E[e^{tY}]$.

c. (4 points) From the definition of moment generating function, write down a series for $m(t)$.

Solution.

$$m(t) = E[e^{tY}] = \sum_{y=0}^{\infty} e^{ty} e^{-\lambda}\frac{\lambda^y}{y!}$$

d. (8 points) Sum the series and give an expression for $m(t)$.

Solution.

$$\sum_{y=0}^{\infty} e^{ty} e^{-\lambda}\frac{\lambda^y}{y!} = e^{-\lambda}\sum_{y=0}^{\infty}\frac{(\lambda e^t)^y}{y!} = e^{-\lambda}e^{\lambda e^t} = e^{\lambda(e^t-1)}$$

9. (20 points) Let Y be a random variable with probability function given by $p(-1) = \frac{1}{8}$, $p(0) = \frac{3}{4}$, and $p(1) = \frac{1}{8}$.

a. (1 point) Find $E[Y]$.

Solution. 0.

b. (4 points) Find $V[Y]$.

Solution. 1/4.

c. (6 points) One part of Tchebysheff's Theorem gives an inequality for $P(|Y-\mu| < k\sigma)$. State the *other* part of Tchebysheff's Theorem (the other inequality).

Solution. $P(|Y-\mu| \geq k\sigma) \leq \frac{1}{k^2}$.

d. (11 points) Let $k = 2$ and Y be the random variable of this problem. Find both sides of the inequality you gave in (c).

Solution. The two sides are both 1/4 . Note that if the $\geq$ inequality in (c) were $>$ instead, we would have $0 \leq 1/4$, which is still true, but not "best possible".

10. (6 points) Let a be a real number and Y a random variable. The variances $V[Y]$ and $V[aY]$ are related by which of the following equations:

(a) $V[aY] = |a|V[Y]$
(b) $V[aY] = V[Y]$
(c) $V[aY] = aV[Y]$
(d) $V[aY] = a^2V[Y]$

Solution. (d).

11. (6 points) Let X and Y be random variables and consider the following propositions:

(1) $E[X+Y] = E[X] + E[Y]$
(2) $V[X+Y] = V[X] + V[Y]$

Which statement most precisely describes the conditions required for the truth of the above propositions?

(a) (2) is true in general; (1) is true if X and Y are independent
(b) (1) is true in general; (2) is true if X and Y are independent
(c) both are true
(d) both are true if X and Y are independent

Solution. (b).