Math 447 - February 8, 2013 - Test 1 Solutions

Name:______________________________

Read these instructions carefully: The points assigned are *not* meant to be a guide to the difficulty of the problems. If the question is multiple choice, there is a penalty for wrong answers, so that your expected score from guessing at random is zero. No partial credit is possible on multiple-choice and other no-work-required questions.

1. (25 points) Of the travelers arriving at a small airport, 60% fly on major airlines, 30% fly on privately owned planes, and the remainder fly on commercially owned planes not belonging to a major airline. Of those traveling on major airlines, 50% are traveling for business reasons, whereas 60% of those arriving on private planes and 90% of those arriving on other commercially owned planes are traveling for business reasons.

Suppose that we randomly select one person arriving at this airport, and wish to find the probability that this person arrived on a privately owned plane, given that the person is traveling for business reasons.

a. (5 points) Name the events defined in the problem. Hint: "Let M be the event that ..."

Solution.

- Let M be the event that the traveler arrives on a major airline.
- Let R be the event that the traveler arrives on a privately owned plane.
- Let C be the event that the traveler arrives on a commercially owned plane not belonging to a major airline.
- Let B be the event that the person is traveling for business reasons.
- Write $\bar{B}$ for the event that the person is traveling for reasons other than business.

b. (10 points) State the equation that will allow you to obtain the desired probability, in terms of the notation established in part (a).

Solution. The desired probability is $P(R \mid B)$, which can be obtained from Bayes' Formula:

$$P(R \mid B) = \frac{P(B \mid R)P(R)}{P(B \mid R)P(R) + P(B \mid M)P(M) + P(B \mid C)P(C)}$$

c. (5 points) The numbers in the problem give various probabilities. Write down these probabilities using the notation you established in part (a).

Solution.

- $P(M) = 60\%$
- $P(R) = 30\%$
- $P(C) = 100\% - 60\% - 30\% = 10\%$
- $P(B \mid M) = 50\%$
- $P(B \mid R) = 60\%$
- $P(B \mid C) = 90\%$

d. (5 points) Now solve the problem: find the probability that this person arrived on a privately owned plane, given that the person is traveling for business reasons. *To obtain credit your final answer must be expressed as a percentage—complex fractions are not acceptable!*

Solution. We plug the numbers from part (b) into the equation from part (c).

$$P(R \mid B) = \frac{0.30 \cdot 0.60}{0.50 \cdot 0.60 + 0.60 \cdot 0.30 + 0.90 \cdot 0.10} = \frac{0.18}{0.300.18 + 0.09} = \frac{18}{57} \approx 31.6\% < \frac{1}{3}$$

2. (20 points) A diagnostic test for a disease is such that it (correctly) detects the disease in 90% of the individuals who actually have the disease. Also, if a person does not have the disease, the test will report that he or she does not have it with probability .9. Only 1% of the population has the disease in question. If a person is chosen at random from the population and the diagnostic test indicates that she has the disease, what is the conditional probability that she does, in fact, have the disease? *To obtain credit your final answer must be expressed as a percentage—complex fractions are not acceptable!*

Solution. The answer is approximately 9%, as we will show. First we establish notation and name the events:

- Let T be the event that the person tests positive for the disease.
- Let D be the event that the tested person actually has the disease.

The given information tells us that $P(D) = 1\%$, $P(T \mid D) = 90\%$, and $P(\bar{T} \mid \bar{D}) = 90\%$. From this we deduce that $P(T \mid \bar{D}) = 1 - P(\bar{T} \mid \bar{D}) = 10\%$ and that $P(\bar{D}) = 99\%$.

We wish to find $P(D \mid T)$, so we apply Bayes' Formula and plug in the numbers of the previous paragraph:

$$P(D \mid T) = \frac{P(T \mid D)P(D)}{P(T \mid D)P(D) + P(T \mid \bar{D})P(\bar{D})} = \frac{0.9 \cdot 0.01}{0.9 \cdot 0.01 + 0.1 \cdot 0.99} = \frac{0.009}{0.009 + 0.099} = \frac{1}{12} = 8.33\%$$

3. (40 points) (Monty Hall) In this version of the game you are shown 6 cards face-down, of which two are red queens, and four are black spot cards. The objective is to select a queen. After you make your initial selection, the host picks out another card (not your selection), and turns it over to show you it is black. (He can always do this because he knows where the red queens are.)

He then offers you the opportunity to change your choice to one of the four remaining cards. (The ones which are neither your original selection nor the black card he showed you.)

a. (5 points) Establish notation: name the relevant events.

Solution.

- Let Q be the event that the initial choice of card is a queen.
- Let B be the event that the initial choice of card is a black spot card.
- Let W be the event that the final choice of card (after the switch) is a queen.

b. (5 points) What is the probability of winning by sticking with your original choice?

Solution.

$$P(Q) = 2/6 = 1/3$$

c. (15 points) What is the probability of winning by switching your choice?

Solution. We apply the "Law of Total Probability":

$$P(W) = P(W \mid Q)P(Q) + P(W \mid B)P(B)$$

We know that if the initial choice of card is a queen, after we are shown a black card there is one queen among the four remaining cards. Thus $P(W \mid Q) = 1/4$. Similarly,

$P(W \mid B) = 2/4 = 1/2$. We computed $P(Q)$ in (a) and $P(B) = 1 - P(Q) = 2/3$. Thus we have

$$P(W) = \frac{1}{4} \cdot \frac{1}{3} + \frac{2}{3} \cdot \frac{1}{2} = \frac{5}{12}.$$

d. (15 points) The prize for picking out a queen is \$100. After showing you the black card, but before you have made the choice to switch or to stay with your original choice, the host makes an offer: For \$15, you can see another black card before deciding whether to switch. Should you accept this offer and pay \$15? Why?

Solution. The offer should be refused. As in (c) we apply the "Law of Total Probability":

$$P(W) = P(W \mid Q)P(Q) + P(W \mid B)P(B),$$

but this time we have $P(W \mid Q) = 1/3$ and $P(W \mid B) = 2/3$. As before we compute

$$P(W) = \frac{1}{3} \cdot \frac{1}{3} + \frac{2}{3} \cdot \frac{2}{3} = \frac{5}{9}.$$

Our expected gain by paying to see the extra card, in dollars, is

$$100 \cdot (\frac{5}{9} - \frac{5}{12}) - 15 = 100 \cdot \frac{60 - 45}{108} - 15 = (\frac{100}{108} \cdot 15) - 15 < 0.$$

Since this is negative, we should refuse the offer.

4. (9 points) The release of two out of three prisoners has been announced. but their identity is kept secret. One of the prisoners considers asking a friendly guard to tell him who is the prisoner other than himself that will be released, but hesitates based on the following rationale: at the prisoner's present state of knowledge, the probability of being released is 2/3, but after he knows the answer, the probability of being released will become 1/2, since there will be two prisoners (including himself) whose fate is unknown and exactly one of the two will be released.

Which of the following statements best describes the prisoner's reasoning and conclusion about the probability (p) of release after inquiring with the guard?

(a) The conclusion $p = 1/2$ is correct, but the reasoning is incorrect.
(b) The conclusion $p = 1/2$ is incorrect, and the reasoning is also incorrect.
(c) The conclusion $p = 1/2$ is correct, and the reasoning is correct also.
(d) The reasoning is partly right, and a corrected version of it shows that $p = 1/3$.

Solution. (b). This is the original Monty Hall problem in disguise. The prisoner's probability of release is unchanged by his acquisition of the information. This should be clear, since neither the prisoner nor the guard have anything to do with the decision, which was made in advance of their conversation anyway. A case-by-case analysis as in the original problem will produce the answer 2/3, and you may wish to try this on your own.

5. (16 points) Let A and B be independent events with $P(A) = 0.4$ and $P(B) = 0.25$.

a. (6 points) What is the definition of independence? (Give the equation that must be satisfied for A and B to be independent.)

Solution. Events A and B are independent if $P(A \cap B) = P(A)P(B)$. (Other versions of the required equality, in terms of conditional probability, are also acceptable.)

b. (5 points) Find $P(A \cup B)$

Solution. Note that $P(A \cap B) = P(A)P(B) = 0.4 \cdot 0.25 = 0.1$ by independence, so $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.55$.

c. (5 points) Find $P(\bar{A} \cap \bar{B})$

Solution. By DeMorgan's law, $\bar{A} \cap \bar{B} = \overline{(A \cup B)}$, so $P(\bar{A} \cap \bar{B}) = 1 - P(A \cup B) = 1 - 0.55 = 0.45$ by (a).

6. (15 points) Vlad is to play a 2-game chess match with Gary and wishes to maximize his chances of winning (and minimize the chances of Gary winning). To do this, he may select a strategy right before he plays each game: timid or bold.

Unfortunately, Gary is the superior player. If Vlad plays timidly, Gary will still win 10 percent of those games, and the rest will be draws. If Vlad plays boldly, Gary will win 5/9 of those games, and lose the rest.

a. (6 points) Describe Vlad's optimal strategy in this 2-game match.

Solution. Vlad should play boldly in the first game, and if he wins, play timidly in the second game. If he loses the first game, he should play boldly in the second game to maximize the odds of a tied match.

b. (9 points) Which of the following best describes the possible outcomes of a match between Vlad and Gary? (Assume Vlad uses his optimal strategy for the match in question.)

(a) Vlad is more likely to win the 2-game match than Gary. He would also be a favorite in a 50-game match, because he can vary his style of play.

(b) Gary is the superior player, and so is more likely to win the 2-game match, and also more likely to win a 50-game match.

(c) Gary is the favorite in the 2-game match, but in the long run Vlad's ability to vary his style of play will give him the advantage. Vlad is the favorite in a 50-game match.

(d) Vlad is more likely to win the 2-game match than Gary, but is an underdog in a 50-game match.

Solution. (d). The match strategy described helps more in a short match than a long one. To understand why, count a win for Vlad as 1, a draw as 1/2, and a loss as 0. One sees from the given conditions that Vlad's expected score is less than 45% of the possible points. The way Vlad becomes the favorite is by sacrificing the possibility of winning 2-0 to hold the lead after the first game. In the longer match, Gary has so many games to catch up that this idea will no longer work.

7. (15 points) Suppose that there is a 1 in 10 chance of injury on a single skydiving attempt.

a. (6 points) If we assume that the outcomes of different jumps are independent, what is the probability that a skydiver is injured if she jumps twice?

Solution. Let J_1 be the event that the skydiver is injured on the first jump and J_2 the event she is injured on the second. The problem asks for

$$P(J_1 \cup J_2) = P(J_1) + P(J_2) - P(J_1 \cap J_2) = P(J_1) + P(J_2) - P(J_1) \cdot P(J_2) = 0.1 + 0.1 - 0.01 = 0.19.$$

b. (9 points) A friend claims if there is a 1 in 10 chance of injury on a single jump then there is a 100% chance of injury if a skydiver jumps 10 times. Is your friend correct? What, approximately, is the probability of injury in 10 jumps?

Solution. No, the friend is not correct. Let T be the event that the skydiver is injured in one of the ten jumps. Then $\bar{T}$ is the event that the skydiver makes 10 jumps without injury. Observe that $\bar{T} = \bar{J}_1 \cap \cdots \cap \bar{J}_{10}$, and since the jumps are independent, $P(\bar{T}) = (1 - \frac{1}{10})^{10}$. Since $\lim_{n\to\infty}(1 - \frac{1}{n})^n = 1/e \approx 0.36$, we have

$$P(T) = 1 - P(\bar{T}) \approx 1 - \frac{1}{e} \approx 1 - 0.36 \approx 0.64.$$