Math 447 - April 23, 2013 - Test 3 Solutions

Name:______________________________

Read these instructions carefully: The points assigned are *not* meant to be a guide to the difficulty of the problems. If the question is multiple choice, there is a penalty for wrong answers, so that your expected score from guessing at random is zero. No partial credit is possible on multiple-choice and other no-work-required questions.

1. (6 points) A bag contains 99 normal (fair) coins and one gaffed coin which has two heads. A single coin is selected at random from the bag and flipped 6 times. It comes up heads every time. The chance that it is actually the two-headed coin is closest to:

(a) 0.20
(b) 0.40
(c) 0.60
(d) 0.80

Solution. (b). Let G be the event that the selected coin is the two-headed coin and let R be the event that the selected coin comes up heads 6 times in a row. Then we have $P(G) = 0.01$, $P(\bar{G}) = 0.99$, $P(R \mid G) = 1$, and $P(R \mid \bar{G}) = 2^{-6}$. By Bayes' Formula we have:

$$P(G \mid R) = \frac{P(R \mid G)P(G)}{P(R \mid G)P(G) + P(R \mid \bar{G})P(\bar{G})} = \frac{1 \cdot 0.01}{1 \cdot 0.01 + 2^{-6} \cdot 0.99} \approx 0.39.$$

So the answer is (b).

2. (6 points) A woman has two children, one of whom is a boy born on a Tuesday. Select from (a)-(d) below the answer closest to the probability that both children are boys.

If you need the event and sample space specified more precisely to answer, here is a description of the process for selecting a random woman with two children and the attendant assumptions: We gather all those women in the world who have exactly two children, tell each of them to "go home unless you have a boy born on a Tuesday", and select a woman randomly from those who remain. Assume that births are equally likely to occur on any day of the week, and that on any given day, boys and girls are equally likely.

(a) 0.51
(b) 0.48
(c) 0.34
(d) 0.26

Solution. (b). We classify children into 14 equiprobable types: they can be boys or girls, and they can be born on any day of the week. Under the assumptions of the problem, women with two children are equally likely to have any of the $196 = 14^2$ possible types of ordered pairs of children. If one of the two is a boy born on a Tuesday, we know that the mother has one of the 27 ($= 14 + 14 - 1$) possible ordered pairs of child-types which contain at least one boy born on a Tuesday. Of these 27 pairs of child-types, 13 contain another boy. Thus the answer is $13/27 \approx 0.48$.

3. (6 points) Before going on vacation for a week, you ask your absent-minded friend to water your ailing plant. Without water, the plant has a 90 percent chance of dying. Even with proper watering, it has a 20 percent chance of dying. And the probability that your friend will forget to water it is 30 percent.

If your friend forgets to water the plant, the probability it will be dead when you return is closest to:

(a) 0.93
(b) 0.83
(c) 0.73
(d) 0.63

Solution. You are given in the problem that this probability is 90%, so the answer is (a).

4. (6 points) *(Exactly the same story and numbers as the previous problem, but a slightly different question.)* Before going on vacation for a week, you ask your absent-minded friend to water your ailing plant. Without water, the plant has a 90 percent chance of dying. Even with proper watering, it has a 20 percent chance of dying. And the probability that your friend will forget to water it is 30 percent.

If the plant is dead when you return, the probability that your friend forgot to water it is closest to:

(a) 0.62
(b) 0.72
(c) 0.82
(d) 0.92

Solution. (a). In this version, you do have to apply Bayes' Formula. Let D be the event that the plant is dead on your return and let F be the event that your friend forgets to water it. The information in the problem statement is: $P(D \mid F) = 0.9$, $P(D \mid \bar{F}) = 0.2$, and $P(F) = 0.3$. By Bayes' Formula we have:

$$P(F \mid D) = \frac{P(D \mid F)P(F)}{P(D \mid F)P(F) + P(D \mid \bar{F})P(\bar{F})} = \frac{0.9 \cdot 0.3}{0.9 \cdot 0.3 + 0.2 \cdot 0.7} \approx 0.658.$$

So the answer is (a).

5. (6 points) Scores on an examination are normally distributed with mean 78 and standard deviation 6. Which of the following numbers is closest to the probability that a student's score exceeds 84, given that it exceeds 72?

(a) 0.14
(b) 0.16
(c) 0.18
(d) 0.21

Solution. (c). The problem asks for $P(X > 84 \mid X > 72)$, and this is $P(X > 84)/P(X > 72)$, by definition of conditional probability. (Note that we do not need to take an intersection, as $X > 84$ implies $X > 72$.) Both of these probabilities can be obtained from the table of the normal distribution, using the fact that $Z = (X-78)/6$ is a standard normal random variable. We find $P(Z > 1) = 0.1587$ (look it up) and $P(Z > -1) = 1-P(Z > 1) = 1-0.1587$ by the symmetry of the normal distribution. So the probability is $0.1587/(1-0.1587) \approx 0.1886$ and the answer is (c).

6. (6 points) Scores on an examination are normally distributed with mean 78 and standard deviation 6.

Which of the following numbers is closest to the proportion of students who have scores 5 or more points above the score that cuts off the lowest 25%?

(a) 0.45
(b) 0.53
(c) 0.60
(d) 0.68

Solution. (a). First find the score that cuts off the lowest 25%. For a standard normal random variable Z, we have $P(Z > 0.67) \approx 0.25$. By symmetry, $P(Z < -0.67) \approx 0.25$. If X is normal with mean 78 and variance 36, then $P(X > 78 - 0.67 \cdot 6) \approx 0.25$. Thus the cutoff score is about 74, and 5 points above this is 79. This is greater than the mean 78, and so has probability less than 1/2. So the answer is (a).

7. (9 points) Suppose that a random variable Y has a probability density function given by

$$f(y) = \begin{cases} ky^3 e^{-y/2}, & y > 0 \\ 0, & \text{elsewhere.} \end{cases}$$

a. (3 points) The value of k that makes f a probability density function is closest to

(a) 0.01
(b) 0.5
(c) 1
(d) 12

Solution. (a). Using out knowledge of the gamma distribution, k must be $1/(\beta^\alpha \Gamma(\alpha))$, and we have $\alpha = 4$, $\beta = 2$. Thus $k = 1/96$ and the answer is (a).

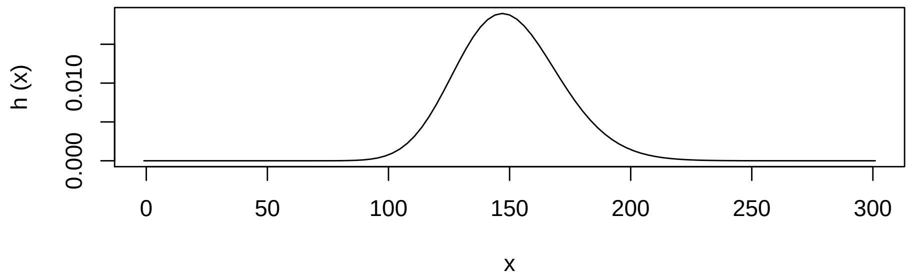
b. (6 points) The value of $E[Y^2]$ is closest to

(a) 63
(b) 101
(c) 109
(d) 120

Solution. (a). Again using our knowledge of the gamma distribution, we have $E[Y^2] = V[Y] + E[Y]^2 = \alpha\beta^2 + (\alpha\beta)^2 = 4 \cdot 2^2 + 4^2 \cdot 2^2 = 80$.

8. (12 points) Let Z be a standard normal random variable. Find, with proof, $E[Z^4]$.

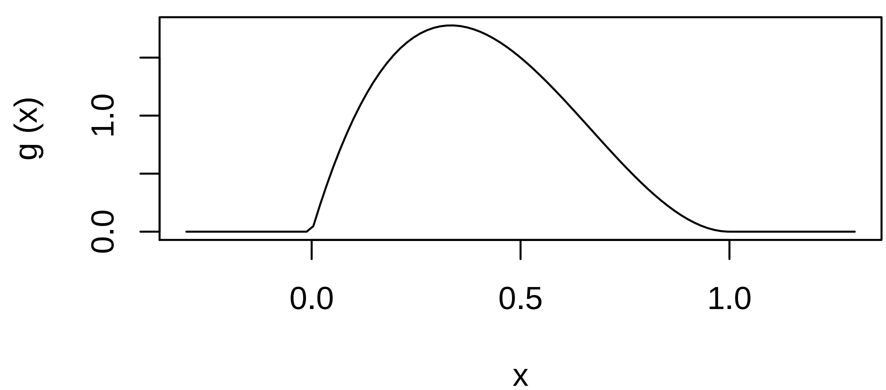
Solution. The answer is 3. This is treated in Exercise 4.199, discussed in class. Alternatively, you could differentiate the moment generating function 4 times, or use the fact that Z^2 is a $\chi^2[1]$ random variable with known mean and variance.



9. (4 points) From which of the following distributions does the probability density function graphed above arise?

(b) the beta distribution with parameters $\alpha = 1$ and $\beta = 2$
(b) the normal distribution with parameters $\mu = 150$ and $\sigma = 5$
(c) the gamma distribution with parameters $\alpha = 50$ and $\beta = 3$
(d) the chi-square distribution with 100 degrees of freedom
(e) none of the above

Solution. (c). The gamma distribution with parameters $\alpha = 50$ and $\beta = 3$ has mean 150, standard deviation $\sqrt{450} \approx 21$, and is approximately normal. You know this, because you looked at various gamma distributions using the "applet exercises".



10. (4 points) From which of the following distributions does the probability density function graphed above arise?

(a) the beta distribution with parameters $\alpha = 3$ and $\beta = 3$
(b) the beta distribution with parameters $\alpha = 5$ and $\beta = 5$

(c) the normal distribution with parameters $\mu = 0.3$ and $\sigma = 0.5$

(d) the beta distribution with parameters $\alpha = 2$ and $\beta = 3$

(e) none of the above

Solution. (d). The first three are symmetric, and if you know the mean, variance, and support of the beta distribution, you see that (d) is a good match.

11. (32 points) Let X and Y be random variables representing the coordinates of a point which is chosen at random from the triangle in the (x, y) plane with vertices $(-1, 0)$, $(1, 0)$, and $(0, 1)$.

a. (4 points) Write down a formula for the probability density function $f(x, y)$.

Solution. We are given that the distribution is uniform over the triangle, so $f(x, y)$ is constant (c) inside the triangle and 0 outside. Since $\int\int f = 1$, c is the inverse of the area of the triangle, which happens to be 1. Thus

$$f(x, y) = \begin{cases} 1, & y \geq 0, x + y \leq 1, y - x \leq 1 \\ 0, & \text{elsewhere.} \end{cases}$$

b. (4 points) Give the *definition* of the marginal density function $f_X(x)$.

Solution.

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y) dy.$$

c. (6 points) Find $f_X(x)$. Splitting into cases if necessary, be sure your formula is valid for all real values of x.

Solution.

$$f_X(x) = \begin{cases} 1 + x, & -1 \leq x \leq 0 \\ 1 - x, & 0 \leq x \leq 1 \\ 0, & \text{elsewhere.} \end{cases}$$

d. (4 points) Give the *definition* of the conditional density function $f(y \mid X = x)$.

Solution.

$$f(y \mid X = x) = \frac{f(x, y)}{f_X(x)}$$

e. (4 points) Give the *definition* of $\text{Cov}(X, Y)$.

Solution. $\text{Cov}(X, Y) = E[(X - \mu)(Y - \nu)]$, where $\mu = E[X]$ and $\nu = E[Y]$.

f. (10 points) Find $\text{Cov}(X, Y)$.

Solution. Note that $\text{Cov}(X, Y) = E[XY] - \mu \cdot \nu$, where $\mu = E[X]$ and $\nu = E[Y]$. Using $f_X(x)$, or symmetry, we get $\mu = 0$, so the second term is zero. To obtain

$$E[XY] = \int\int xyf(x, y) dx dy$$

we can either use symmetry or do the integration directly. The answer, either way, is zero. Thus $\text{Cov}(X,Y) = 0$.

g. (4 points) Are X and Y independent? (Only an answer is necessary, no reasoning required.)

Solution. No, they are not.

12. (6 points) Suppose the moment generating function of Y is $1/(1-3t)$. What is the distribution of Y? Indicate the values of the parameters associated with this distribution.

Solution. Y has the gamma distribution with parameters $\alpha = 1$ and $\beta = 3$. (This is the same as the exponential distribution, so credit for that answer also.)

13. (10 points) Suppose that the random variables X and Y are such that $E[X] = 4$, $E[Y] = -1$, $V[X] = 2$, and $V[Y] = 8$.

a. (2 points) Find $\text{Cov}(X,X)$.

Solution. $\text{Cov}(X,X) = V[X] = 2$.

b. (8 points) What is the largest possible value of $\text{Cov}(X,Y)$?

Solution. We know that $\text{Cov}(X,Y) = \rho\sigma_X\sigma_Y$, and $\rho \leq 1$. So $\text{Cov}(X,Y) \leq \sqrt{2} \cdot \sqrt{8} = 4$.

14. (12 points) Suppose that W is normal with mean 3 and variance 1. Suppose X has the gamma distribution with parameters $\alpha = 2$ and $\beta = 1$. Let $Y = 2W - 1$ and $Z = 2X - 1$.

a. (6 points) Which of the following statements about the random variables Y and Z is most accurate?

(a) Y is normally distributed and Z has the gamma distribution.

(b) Y is not normally distributed but Z has the gamma distribution.

(c) Y is not normally distributed and Z does not have the gamma distribution.

(d) Y is normally distributed but Z does not have the gamma distribution.

Solution. (d). See the explanation in the next question.

b. (6 points) Prove that your answer in part (a) is correct.

Solution. Note that the moment-generating function of $Y = 2W - 1$ is $e^{-1t}m(2t)$, where $m(t)$ is the moment-generating function of W. Since the moment-generating function of W is $\exp(3t + t^2/2)$, the moment-generating function of Y is $e^{-t}\exp(3(2t) + (2t)^2/2) = exp(5t + 2^2t^2/2)$, which is the moment generating function of a normal random variable with mean 5 and standard deviation 2. By uniqueness of moment generating functions, Y is normal.

Since X has the gamma distribution, it can take on any positive value. But $Z = 2X - 1$ can take on negative values. Thus Z does not have the gamma distribution, which only takes on nonnegative values.