# Chapter 6: Functions of Random Variables

We are often interested in a function of one or several random variables, $U(Y_1, \ldots, Y_n)$.

We will study three methods for determining the distribution of a function of a random variable:

1. The method of cdf's
2. The method of transformations
3. The method of mgf's

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# The method of cdf's

For a function $U$ of a continuous r.v.'s $Y$ with a known density $f(y)$, we can often find the distribution of $U$ using the definition of the cdf.

Example: Let $Y$ be the amount of sugar produced per day (in tons). Suppose

$$f(y) = \begin{cases} 2y, & \text{for } 0 < y < 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Let the profit $U = 3Y - 1$ (in hundreds of \$).
Then, $F_U(u) =$ and $f_U(u) =$.

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# Example

Let $Y$ have the pdf:

$$f(y) = \begin{cases} 6y(1-y), & \text{for } 0 < y < 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Find the pdf of $U = Y^3$.

# Bivariate situation

Let $U = h(X, Y)$ be a function of r.v.'s $X$ and $Y$ that have joint density $f(x, y)$.

We must:

- Find the values $(x, y)$ such that $U \leq u$.
- Integrate $f(x, y)$ over this region to obtain $\mathbb{P}(U \leq u) = F_U(u)$.
- Differentiate $F_U(u)$ to obtain $f_U(u)$.

Example Let $X$ be the amount of gasoline stocked at the beginning of the week. Let $Y$ be the amount of gasoline sold during the week. The joint density of $X$ and $Y$ is

$$f(x, y) = \begin{cases} 3x, & \text{for } 0 \leq y \leq x \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Find the density of $U = X - Y$, the amount of gasoline remaining at the end of the week.

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# Example

Suppose the joint density of $X$ and $Y$ is

$$f(x, y) = \begin{cases} 6e^{-3x-2y}, & \text{for } 0 \leq x, 0 \leq y, \\ 0, & \text{elsewhere.} \end{cases}$$

Find the pdf of $U = X + Y$.

# Quiz

Let $X$ be an exponential variable with parameter $\beta = 1$. Let $U = 1/X$.
What is the value of the density of $U$ at $u = \frac{1}{2}$?

(A) $e^{-2}$
(B) $4e^{-2}$
(C) $e^{-1/2}$
(D) $\frac{1}{2}e^{-1/2}$
(E) other

# Another example

Suppose the joint density of $X$ and $Y$ is

$$f(x,y) = \begin{cases} 5xe^{-xy}, & \text{for } 0.2 \le x \le 0.4, 0 \le y, \\ 0, & \text{elsewhere.} \end{cases}$$

Find the pdf of $U = XY$.

# Application: how do we generate a random variable ?

Suppose $X$ is a continuous random variable with density $f(x)$. How can we sample $X$?

Compute cdf $F(x)$ and its functional inverse $F^{(-1)}(u)$, that is, the quantile function.

Let $U \sim Unif(0,1)$. Consider $Y = F^{(-1)}(U)$. What is the distribution of $Y$?

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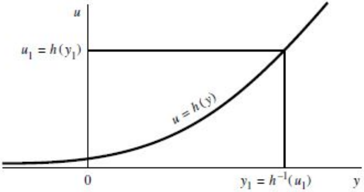
# The Method of Transformations

This method can be used to find the pdf of $U = h(Y)$ if $h$ is a one-to-one function (for example, an increasing or a decreasing function).




If so, the inverse function $h^{-1}(u)$ exists.

$$f_U(u) = f_Y(h^{-1}(u)) \left| \frac{h^{-1}(u)}{du} \right| = f_Y(h^{-1}(u)) \left| \frac{1}{h'\left(h^{-1}(u)\right)} \right| .$$

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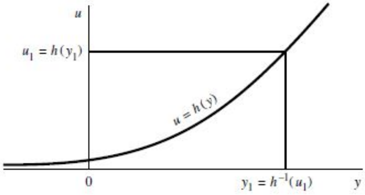


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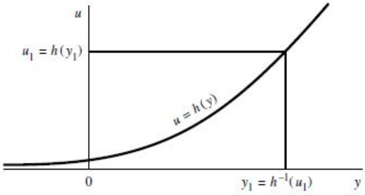


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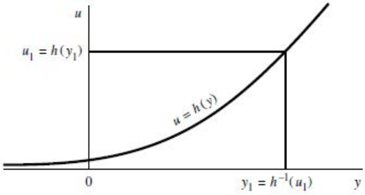


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# Proof

Suppose $h(u)$ is an increasing function. Then

$$\{y : h(y) \leq u\} \equiv \{y : y \leq h^{-1}(u))\}.$$

It follows that $F_U(u) = F_Y(h^{-1}(u))$.

By using the chain rule for differentiation, we find the stated formula.
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# The Method of Transformations: Steps

1. Verify $h(\cdot)$ is and increasing function or a decreasing function.
2. Write the inverse function $y = h^{-1}(u)$.
3. Find the derivative $\frac{dh^{-1}(u)}{du}$ of this inverse function.
4. Then the density of $U = h(Y)$ is: $f_U(u) = f_Y(h^{-1}(u)) \left| \frac{dh^{-1}(u)}{du} \right|$.

Example Let $Y$ be a standard exponential r.v. Find the pdf of $U = \sqrt{Y}$.

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# Example

Let $U \sim \textit{unif}(-\pi/2, \pi/2)$. Find the pdf of $X = \textit{tan}(U)$.

Note $h(u) = \tan(u)$ is monotone on $(-\pi/2, \pi/2)$. The inverse function is ...

The distribution of $X$ is called a standard Cauchy distribution. It is remarkable for its property that $\mathbb{E}(X)$ does not exists.

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# Example

Let $Z \sim N(0, 1)$. Find the pdf of $U = Z^2$.

Problem: $h(z) = z^2$ is not monotone on $(-\infty, \infty)$.

**Solution**: The inverse function $h^{-1}(u) = \sqrt{u}$ is multivalued. It has two values for each $u$: positive and negative values of the square root.

To get the pdf of $U$, we can add together the contributions from both 'branches' of the inverse function.

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# Bivariate Transformation Technique

What if we have a bivariate transformation of $X$ and $Y$? Suppose

$$U = g(X, Y),$$

$$V = h(X, Y),$$

where the transformation is one-to-one.
Let the inverse functions be denoted by x(u,v) and y(u,v).

If $f(x, y)$ is the joint pdf of $(X, Y)$, then the joint pdf of $(U, V)$ is:

$$f(x(u, v), y(u, v)) \left| \det \begin{pmatrix} \frac{\partial x(u,v)}{\partial u} & \frac{\partial x(u,v}{\partial v} \\ \frac{\partial y(u,v)}{\partial u} & \frac{\partial y(u,v)}{\partial v} \end{pmatrix} \right|$$

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# Example

Suppose $X$ and $Y$ are r.v.'s with joint density

$$f(x, y) = \begin{cases} e^{-(x+y)}, & \text{if } x > 0, y > 0, \\ 0, & \text{elsewhere.} \end{cases}$$

Find the joint density of $U = X + Y$ and $V = \frac{X}{X+Y}$.

Find the marginal density of $V$.

First, solve for inverse functions.

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Find the density of $U = XY$.

Define $V = X$. Solve for the inverse transformation from $U, V$ to $XY$.

After determining the joint density of $U$ and $V$, find the marginal density of $U$ by integrating out $V$.

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# Method of moment-generating functions

If $X$ and $Y$ are two random variables with the same moment generating function, then $X$ and $Y$ have the same probability distribution.

Suppose $U = U(Y_1, \ldots, Y_n)$, and we can find the mgf of $U$, $m_U(t)$.

If we recognize $m_U(t)$ as the mgf of a known distribution, then $U$ has that distribution.

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# Example

Suppose $Y \sim \textit{gamma}(\alpha, \beta)$. What is the distribution of $U = 2Y/\beta$?

We know that the mgf of $Y$ is

$$m_Y(t) = \frac{1}{(1-\beta t)^\alpha}.$$

So, the mgf of $U$ is

$$m_U(t) = \ldots$$

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# Mgf Method

The method of mgf's is especially useful for deriving the distribution of the sum of independent random variables

If $X_1, \ldots, X_n$ are independent and $U = X_1 + \ldots + X_n$, then

$$m_U(t) = \prod_{i=1}^{n} m_{X_i}(t).$$

**Example:** Suppose $X_1, \ldots, X_n$ are independent $\mathrm{Expon}(\beta)$. What is the distribution of $U = X_1 + \ldots + X_n$?

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**Example:** Suppose $X_1, \ldots, X_n$ are independent Poisson r.v.'s with parameters $\lambda_1, \ldots, \lambda_n$. What is the distribution of $U = X_1 + \ldots + X_n$?

# Mgf Method

**Example**: Suppose $X_1, \ldots, X_n$ are independent normal r.v.'s with means $\mu_1, \ldots, \mu_n$ and variances $\sigma_1^2, \ldots \sigma_n^2$. What is the distribution of $U = X_1 + X_2 + \ldots + X_n$?

# Mgf Method: Quiz

Suppose $X_1, \ldots, X_n$ are independent r.v.'s with distribution $\Gamma(\alpha, \beta)$. What is the distribution of $U = X_1 + X_2 + \ldots + X_n$?

(A) $\Gamma(n\alpha, \beta)$
(B) $\Gamma(\alpha, n\beta)$
(C) Other

# Mgf Method: Examples

Example: Suppose $Z_1, \dots, Z_n$ are independent standard normal r.v.'s. What is the distribution of $U = Z_1^2 + Z_2^2 + \dots + Z_n^2$?

# The Central Limit Theorem

Setting:
Let $X_1, X_2, \ldots$ be i.i.d. random variables all with mean $\mu$ and standard deviation $\sigma$. Let

$$\overline{X}_n = \frac{X_1 + X_2 + ... + X_n}{n} = \frac{1}{n}\sum_{i=1}^{n} X_i.$$

Conclusion: For large $n$:

$$\overline{X}_n \approx N(\mu, \sigma^2/n)$$

Standardized $\overline{X}_n$ is distributed approximately as $N(0, 1)$.

The CLT tells us that the sampling distribution of $\overline{X}_n$ is approximately normal as long as the sample size is large.

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# Central Limit Theorem

**Theorem** (CLT or de Moivre's Theorem)
Let $X_1, \ldots, X_n$ be i.i.d random variables with mean $\mu$ and standard deviation $\sigma$.
Then the cdf of $\frac{\overline{X}-\mu}{\sigma/\sqrt{n}}$ converges to the cdf of a standard normal variable:

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt$$

In other words,

$$\mathbb{P}\left(\frac{|\overline{X}-\mu|}{\sigma/\sqrt{n}} \leq x\right) \to \mathbb{P}(Z \leq x),$$

when $n \to \infty$.

# The Central Limit Theorem: Example

The service times for customers at a cashier are i.i.d random variables with mean 2.5 minutes and standard deviation 2 minutes. Approximately, what is the probability that the cashier will take more than 4 hours to serve 100 people?

# Example

To head the newly formed US Dept. of Statistics, suppose that 50% of the population supports Erika, 25% supports Ruthi, and the remaining 25% is split evenly between Peter, Jon and Jerry.

A poll asks 400 random people who they support.
What is the probability that at least 55% of those polled prefer Erika?

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# Order Statistics

The order statistics of a random sample $X_1, \ldots, X_n$ are:
$X_{(1)} =$ the smallest value in the sample,
$X_{(2)} =$ the second-smallest value in the sample,
$\vdots$
$X_{(n)} =$ the largest value in the sample.

So, $X_{(1)} \leq X_{(2)} \leq \ldots \leq X_{(n)}$.

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1. If $X_1, \ldots, X_t, \ldots X_n$ denote the volume of Internet traffic at time $t$. We are interested in behavior of $X_{(n)}$ the maximum traffic volume that occurs in the interval of $n$ minutes.

2. We need a net to hold particles. If If $X_1, \ldots, X_n$ are the sizes of particles, we are interested in $X_{(1)}$, their minimum size.

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# Order Statistics

Some common summary statistics are functions of order statistics.

Sample median

Sample range

The distribution of these statistics depends on the distribution of order statistics.

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Suppose $X_1, \ldots, X_n$ are i.i.d. with pdf $f(x)$ and cdf $F(x)$. What is the pdf for the maximum $X_{(n)}$?

Use the method of cdf's.

$$P(X_{(n)} \leq x) = F(x)^n.$$

So the density is

$$P(X_{(n)} \leq x) = nf(x)F(x)^{n-1}.$$

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What is the pdf for the minimum $Y_{(1)}$?

Examples:
1. Suppose wave heights have an exponential distribution with mean height 10 feet. If 200 waves crash during the night, what is the distribution of the highest wave?

What is the probability that the highest wave is more than 50 feet? more than 100 feet?

2. Suppose light bulbs' lifelengths have an exponential distribution with mean 1200 hours. Two bulbs are installed at the same time. What is the expected time until one bulb has burned out?

What if we had installed 3 lamps?

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# Order Statistics: General case

If $X_1, \ldots, X_n$ are i.i.d continuous r.v.'s with pdf $f(x)$ and cdf $F(x)$, the pdf of the $k$-th order statistic, $X_{(k)}$, is:

$$g_k(x) = \frac{n!}{(k-1)!(n-k)!} F(x)^{k-1} f(x) (1 - F(x))^{n-k}.$$

Proof: The probability $\mathbb{P}(X_{(k)} \leq x)$ is the probability that at least $k$ of $X_i$ are less than or equal to $x$. So,

$$\mathbb{P}(X_{(k)} \leq x) = \sum_{i=k}^{n} \binom{n}{i} F(x)^i (1 - F(x))^{n-i}.$$

One obtain the theorem after differentiation and some algebraic manipulations.

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# Order Statistics: Example

10 numbers are generated at random between 0 and 1. What is the distribution of the 3rd-smallest of these? What is the expected value of the 3rd smallest?

# Order Statistics: Joint distribution

The joint distribution of any two order statistics $X_{(j)}$ and $X_{(k)}$, $j < k$, is

$$\begin{aligned} g_{j,k}(x_j, x_k) = \quad & \frac{\Gamma(n+1)}{\Gamma(j)\Gamma(k-j)\Gamma(n-k+1)} F(x_j)^{j-1} [F(x_k) - F(x_j)]^{k-1-j} \\ & \times [1 - F(x_k)]^{n-k} f(x_j) f(x_k), \text{ if } x_j < x_k, \end{aligned}$$

and 0 elsewhere.

# Review

Classical probability: The probability of an event is equal to the number of favorable cases divided by the number of all possible cases. (Laplace)

Example: Polina and Anton went to a party. There are 6 people at the party and they are all sitted around a circular table in a random order. What is the probability that Polina sits next to Anton.

Axiomatic Probability: Axiom III. To each set A is assigned a non-negative real number P(A). This number is called the probability of the event A. (Kolmogorov)

Examples:

1. Let $P(A) = 0.6$ $P(B) = 0.7$ and $P(A \cap \overline{B}) = 0.1$. What is the probability that one and only one of the events A and B occurs?

2. For some events A and B we know that $P(A|B) = 0.7$, $P(A|\overline{B}) = 0.3$ and $P(B|A) = 0.6$. Calculate $P(A)$.

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Example: From an urn that contains 3 white and 2 black balls, one ball is transferred into another urn that contains 4 white and 4 black balls. Then a ball is drawn from the second urn. Suppose this ball is black. What is the probability that the transferred ball was white?

Example: Three hunters (Polina, Anton, and Natasha) simultaneously shot at a bear, one shot each. The bear was killed with one bullet. What is the probability that the bear was shot by Natasha, if their probability of hit are $p_P = 0.6$, $P_A = 0.4$ and $P_N = 0.2$, respectively?

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# Review

Discrete distributions: Binomial, Geometric, and Poisson.

Continuous distributions: Uniform, Exponential, Normal and others.

Relationship of Poisson and exponential distributions

Example People arrive at a store according to a Poisson process of rate 2 per minute. Let $T_i$ denote the arrival time in minutes of the $i$-th customer.

What is the expected value of the third arrival $T_3$?

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Calculating Expectations, Variances, and Quantiles

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Chebyshev's theorem and its applications.

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Central Limit Theorem

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