

No books, no notes, only SOA-approved calculators. Please put your answers in the spaces provided!

Name: \_\_\_\_\_

Section: \_\_\_\_\_

Question	Points	Score
1	8	
2	6	
3	10	
4	19	
5	9	
6	10	
7	14	
8	14	
9	23	
10	14	
Total:	127	

1. (8 points) (Monty Hall, Again) On this version of the show, you are shown 5 face down cards, one of which is the  $Q\heartsuit$ , one of which is the  $Q\spadesuit$ , and the other three of which are black spot cards. Your objective is to select the  $Q\heartsuit$ , which wins \$1,000. But this time there is a twist: if you wind up with the  $Q\spadesuit$ , you have to pay \$1,000. (You can pay, since you have won a lot of money on previous episodes of the show.) As usual, you have to select a face-down card, and then the host makes an offer to show you a card. After he turns over the card, you will have the opportunity to switch your choice. The offer is as follows: he will turn over the  $Q\spadesuit$  for \$200, or he will turn over a black spot card for \$100. (Note that if you pay the \$200 and your initial selection is the  $Q\spadesuit$ , he will have to show you that card.)

Assuming you wish to maximize your expected winnings (net of any payments made), which statement is most accurate?

- A. The best strategy is to pay \$100 to see a black spot card. Paying \$200 to see the  $Q\spadesuit$  increases your expectation, but is not as favorable as seeing a black spot card.
- B. The best strategy is to pay \$200 to see the  $Q\spadesuit$ . Paying \$100 to see a black spot card increases your expectation, but is not as favorable as seeing the  $Q\spadesuit$ .
- C. The best strategy is to pay \$100 to see a black spot card. Paying \$200 to see the  $Q\spadesuit$  loses money on average.
- D. The best strategy is to pay \$200 to see the  $Q\spadesuit$ . Paying \$100 to see a black spot card loses money on average.**
- E. All of the above statements contain material errors.

**Solution:** Observe that if you select a card at random your expected winnings are zero, as the prize for the  $Q\heartsuit$  is exactly canceled by the negative prize for the  $Q\spadesuit$ .

If the  $Q\spadesuit$  is removed from the list of possibilities, this does help us. Assume that we pay the \$200, see the  $Q\spadesuit$ , and opt to switch. We compute our expected winnings using the “Law of Total Probability”, as usual. Let  $A$  be the event that our initial selection was the  $Q\heartsuit$ ,  $B$  be the event that our initial selection was a spot card, and  $C$  be the event that our initial selection was the  $Q\spadesuit$ . Let  $W$  be the event that our new selection is the winning  $Q\heartsuit$ . Then we have  $P(W | A) = 0$ ,  $P(W | B) = \frac{1}{3}$ , and  $P(W | C) = \frac{1}{4}$ . We compute

$$P(W) = P(W | A)P(A) + P(W | B)P(B) + P(W | C)P(C) = 0 + \frac{1}{3} \cdot \frac{3}{5} + \frac{1}{4} \cdot \frac{1}{5} = \frac{1}{4}$$

Thus, our expected winnings for this strategy are  $\frac{1}{4} \cdot 1000 - 200 = 50 > 0$ .

Similarly, if we pay to see a black spot card, and opt to switch, with events defined as above, we have:  $P(W | A) = 0$ ,  $P(W | B) = 1/4$ , and  $P(W | C) = 1/4$ . If  $X$  is the event that our new selection is the  $Q\spadesuit$ , we have  $P(X | A) = 1/4$ ,  $P(X | B) = 1/4$ , and  $P(X | C) = 0$ . A little arithmetic shows that our expected winnings are zero, minus the \$100 paid to see the card.

Thus the answer is D.

1. \_\_\_\_\_

2. (6 points) As items come to the end of a production line, an inspector chooses which items are to go through a complete inspection. Ten percent of all items produced are defective. Sixty percent of all defective items go through a complete inspection, and 20% of all good items go through a complete inspection. Given that an item is completely inspected, what is the probability it is defective? Choose the answer closest to this probability. A. 0.13 B. 0.19 C. 0.24 D. 0.59 E. 0.64

2. \_\_\_\_\_

**Solution:** Let  $C$  be the event that an item undergoes a complete inspection and  $D$  be the event that it is defective. The problem gives us the following information:  $P(D) = 0.10$ ,  $P(C | D) = 0.60$ ,  $P(\bar{C} | D) = 0.20$ . Now apply Bayes to get  $P(D | C) = 0.25$ , so the answer is C.

3. A soft-drink machine can be regulated so that it discharges an average of  $\mu$  ounces per cup. Ounces of fill are normally distributed with standard deviation  $\sigma$  ounces.

- (a) (5 points) If  $\sigma = 0.3$ , give the setting for  $\mu$  so that 8-ounce cups will overflow only 1% of the time.

(a) \_\_\_\_\_

**Solution:** This is problem 4.75 from the text. Let  $Y =$  volume filled, so that  $Y$  is normal with mean  $\mu$  and  $\sigma = 0.3$ . They require that  $P(Y > 8) = 0.01$ . For the standard normal,  $P(Z > z_0) = 0.01$  when  $z_0 = 2.33$ . Therefore, it must hold that  $2.33 = (8 - \mu)/0.3$ , so  $\mu = 7.301$ .

- (b) (5 points) Assume that the standard deviation  $\sigma$  can be fixed at certain levels by carefully adjusting the machine. What is the largest value of  $\sigma$  that will allow the actual amount dispensed to fall within 1 ounce of the mean with probability at least .95?

(b) \_\_\_\_\_

**Solution:** This is problem 4.76 from the text. It follows that  $0.95 = P(|Y - \mu| < 1) = P(|Z| < 1/\sigma)$ , so that  $1/\sigma = 1.96$  or  $\sigma = 1/1.96 = 0.5102$ .

4. Suppose that  $Y_1$  and  $Y_2$  are random variables with joint density function given by

$$f(y_1, y_2) = \begin{cases} e^{-(y_1+y_2)} & y_1 \geq 0 \text{ and } y_2 \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

- (a) (3 points) The (marginal) distribution of  $Y_1$  is one of the distributions we studied. What is the name of this distribution?

(a) \_\_\_\_\_

**Solution:** Exponential (which is a special case of the gamma distribution).

(b) (2 points) What are the relevant parameters?

(b) \_\_\_\_\_

**Solution:**  $\beta = 1$ .

(c) (9 points) Let  $U = Y_1 + Y_2$ . Find the density function  $f_U(u)$ , using any of the three methods in the text, and put your answer in the space below. No credit will be given if there is no answer in the space given.

$$f_U(u) = \left\{ \begin{array}{l} \text{_____} \\ \text{_____} \end{array} \right.$$

**Solution:** With the method of moment generating functions, we have  $m_U(t) = m_{Y_1}(t)m_{Y_2}(t)$  because of independence. Thus

$$m_U(t) = \frac{1}{1-t} \cdot \frac{1}{1-t} = \frac{1}{(1-t)^2},$$

which we recognize as the mgf of the gamma distribution.

The distribution of  $U$  is one we have studied.

(d) (3 points) What is the name of this distribution?

(d) \_\_\_\_\_

**Solution:** Gamma.

(e) (2 points) What is(are) the relevant parameter(s)?

(e) \_\_\_\_\_

**Solution:**  $\alpha = 2 \beta = 1$

5. An important aspect of a federal economic plan was that consumers would save a substantial portion of the money that they received from an income tax reduction. Suppose that early estimates of the portion of total tax saved, based on a random sampling of 35 economists, had mean 26% and standard deviation 12%.

(a) (7 points) What is the approximate probability that a sample mean estimate, based on a random sample of  $n = 35$  economists, will lie within 1% of the mean of the population of the estimates of all economists?

(a) \_\_\_\_\_

**Solution:** This is problem 7.48 in the text. We are asked to compute  $P(|\bar{Y} - \mu| < 1) = P(-1 < \bar{Y} - \mu < 1)$ , where  $\bar{Y}$  is the sample mean. Because we have a lot of (presumably) independent samples, we may use the central limit theorem, i.e. assume  $\bar{Y}$  is normal. We may also use the sample standard deviation as the population standard deviation, because we have enough samples. By Theorem 7.1 the standard deviation of  $\bar{Y}$  is  $\sigma = 12/\sqrt{35} = 2.028$ . We do not need to know  $\mu$ , only that  $(\bar{Y} - \mu)/\sigma$  is standard normal. So the probability we are interested in is  $P(\frac{-1}{2.028} < Z < \frac{1}{2.028})$ . Compute and use the tables to get 0.3758.

- (b) (2 points) Is it necessarily true that the mean of the population of estimates of all economists is equal to the percent tax saving that will actually be achieved? A. Yes B. No

**Solution:** No, there is no mathematical relationship between the estimates of the economists and the actual results.

(b) \_\_\_\_\_

6. Let  $Y_1$  and  $Y_2$  be independent random variables with identical cumulative distribution functions  $F(y)$  and identical probability density functions  $f(y)$ . Let  $Y = \max(Y_1, Y_2)$ , and write  $G(y)$  for the cumulative distribution function of  $Y$ .

**Solution:** This problem is based on the reasoning in Section 6.7, “Order Statistics”, but simplified from general  $n$  to  $n = 2$ .

- (a) (3 points) Express  $G(y) = P(Y \leq y)$  in terms of  $P(Y_1 \leq y)$  and  $P(Y_2 \leq y)$ .

(a) \_\_\_\_\_

**Solution:** We have  $\max(Y_1, Y_2) \leq y$  iff  $Y_1 \leq y$  and  $Y_2 \leq Y$ . Since  $Y_1$  and  $Y_2$  are independent,

$$P(Y \leq y) = P(Y_1 \leq y) \cdot P(Y_2 \leq y).$$

- (b) (3 points) Express  $G(y)$  in terms of  $F(y)$ .

(b) \_\_\_\_\_

**Solution:** Observe that  $P(Y_1 \leq y) = P(Y_2 \leq y) = F(y)$ , since  $Y_1$  and  $Y_2$  have the same distribution. Thus  $G(y) = F(y)^2$ .

- (c) (4 points) Express the probability density function of  $Y$  in terms of  $F(y)$  and  $f(y)$ .

(c) \_\_\_\_\_

**Solution:** We have  $g(y) = G'(y) = 2F(y)f(y)$ .

7. When commercial aircraft are inspected, wing cracks are reported as nonexistent, detectable, or critical. The history of a particular fleet indicates that 70% of the planes inspected have no wing cracks, 25% have detectable wing cracks, and 5% have critical wing cracks.

(a) (3 points) If 5 planes are randomly selected, find the probability that one has a critical wing crack, two have detectable wing cracks, and two have no wing cracks.

(a) \_\_\_\_\_

**Solution:** Use the theory of the multinomial distribution. The probability is

$$\frac{5!}{1!2!2!}(0.05)^1(0.25)^2(0.7)^2 \approx 0.4594$$

(b) (3 points) Ten planes are randomly selected. Let  $X$  be the number of planes with detectable wing cracks, and  $Y$  be the number of planes with critical wing cracks. The time in hours to repair these planes is  $C = 2X + 10Y$ . Find  $E[C]$ .

(b) \_\_\_\_\_

**Solution:** This requires only the theory of the binomial random variable and linearity of expectation.

$$E[C] = E[2X + 10Y] = 2E[X] + 10E[Y] = 2 \cdot 10 \cdot (0.25) + 10 \cdot 10 \cdot (0.05) = 5 + 5 = 10$$

(c) (3 points) With  $X$  and  $Y$  as in the previous part, find  $\text{Cov}(X, Y)$ .

(c) \_\_\_\_\_

**Solution:** Here you need to know the formula  $\text{Cov}(X, Y) = -npq$  from the theory of the multinomial distribution. We have  $\text{Cov}(X, Y) = 10 \cdot (0.25) \cdot (0.05) = 0.125$ .

(d) (5 points) With  $C$ ,  $X$ , and  $Y$  as in the previous part, find  $V[C]$ .

(d) \_\_\_\_\_

**Solution:** From the definitions  $V[C] = \text{Cov}(C, C)$ , and covariance is bilinear. We know  $V[X]$  and  $V[Y]$  from the formula  $np(1 - p)$  for the binomial distribution, and  $\text{Cov}(X, Y)$  from the previous part.

$$V[C] = \text{Cov}(2X + 10Y, 2X + 10Y) = 2^2 \text{Cov}(X, X) + 2 \cdot 2 \cdot 10 \cdot \text{Cov}(X, Y) + 10^2 \text{Cov}(Y, Y) = 4 \cdot 10 \cdot (0.25)(0.75) + 40 \cdot (0.125) + 100 \cdot 10 \cdot (0.05)(0.95) = 7.5 + 5 + 47.5 = 60$$

8. Let  $Y$  be a random variable with probability density function given by

$$f(y) = \begin{cases} (3/2)y^2 & -1 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) (5 points) Find the cumulative distribution function  $F_Y(y)$ ; put your answer in the form below.

$$f_Y(y) = \begin{cases} \underline{\hspace{2cm}} & \underline{\hspace{2cm}} \\ \underline{\hspace{2cm}} & \underline{\hspace{2cm}} \\ \underline{\hspace{2cm}} & \underline{\hspace{2cm}} \end{cases}$$

**Solution:**

$$F_Y(y) = \int_{-\infty}^y f_Y(y) dy = \begin{cases} 0 & y \leq -1 \\ \frac{1}{2}(y^3 + 1) & -1 < y < 1 \\ 1 & y \geq 1 \end{cases}$$

(b) (9 points) Let  $U = Y^2$  and find the density function  $f_U(u)$ ; put your answer in the form below.

$$f_U(u) = \begin{cases} \underline{\hspace{2cm}} & \underline{\hspace{2cm}} \\ \underline{\hspace{2cm}} & \underline{\hspace{2cm}} \end{cases}$$

**Solution:** First find  $F_U$  and then obtain  $f_U = F'_U(u)$ . We have  $F_U(u) = P(U \leq u) = P(Y^2 \leq u) = P(-\sqrt{u} \leq Y \leq \sqrt{u})$ . This probability is  $F_Y(\sqrt{u}) - F_Y(-\sqrt{u}) = u^{\frac{3}{2}}$  when  $0 < u < 1$ . If  $u \leq 0$  then  $F_U(u) = 0$ , while  $F_U(u) = 1$  if  $u \geq 1$ . Differentiating, we have

$$f_U(u) = \begin{cases} \frac{3}{2}\sqrt{u} & 0 < u < 1 \\ 0 & \text{otherwise.} \end{cases}$$

9. Suppose that you are told to toss a die until you have observed each of the six faces. We consider the number of tosses required to complete your assignment to be a random variable  $Y$ . Write  $Y = Y_1 + Y_2 + Y_3 + Y_4 + Y_5 + Y_6$ , where  $Y_1$  is the trial on which the first face is tossed, (so  $Y_1 = 1$ ),  $Y_2$  is the

number of additional tosses required to get a face different than the first,  $Y_3$  is the number of additional tosses required to get a face different than the first two distinct faces,  $\dots$ ,  $Y_6$  is the number of additional tosses to get the last remaining face after all other faces have been observed. Each  $Y_i$  has a geometric distribution, and the  $Y_i$  are independent.

(a) (2 points) What is the success probability  $p$  (the parameter to the geometric distribution) for  $Y_2$ ?

(a) \_\_\_\_\_

**Solution:**  $\frac{5}{6}$

(b) (2 points) What is the success probability  $p$  (the parameter to the geometric distribution) for  $Y_3$ ?

(b) \_\_\_\_\_

**Solution:**  $\frac{4}{6}$

(c) (3 points) As a function of  $i$ , what is the success probability  $p$  (the parameter to the geometric distribution) for  $Y_i$ ?

(c) \_\_\_\_\_

**Solution:**  $\frac{7-i}{6}$

(d) (2 points) Find  $E[Y_2]$

(d) \_\_\_\_\_

**Solution:**  $E[Y_2] = \frac{1}{p} = \frac{6}{5}$

(e) (2 points) Find  $E[Y_i]$

(e) \_\_\_\_\_

**Solution:**  $E[Y_i] = \frac{1}{p} = \frac{6}{7-i}$

(f) (5 points) Find  $E[Y]$

(f) \_\_\_\_\_

**Solution:**

$$E[Y] = E[Y_1 + \dots + Y_6] = E[Y_1] + \dots + E[Y_6] = 1 + \frac{6}{5} + \frac{6}{4} + \dots + \frac{6}{1} = 14.7$$



(g) (7 points) Find  $V[Y]$

(g) \_\_\_\_\_

**Solution:** Using the independence of the  $Y_i$  (mentioned in the statement of the problem), and the variance of a geometric RV,

$$V[Y] = V[Y_1 + \cdots + Y_6] = V[Y_1] + \cdots + V[Y_6] = 0 + \left(1 - \frac{5}{6}\right) \frac{6^2}{5^2} + \cdots + \left(1 - \frac{1}{6}\right) \frac{6^2}{1^2} = 38.99$$

10. Let  $Y$  be a random variable which has the uniform distribution on the interval  $[0, 1]$ .

(a) (2 points) Find  $\mu = E[Y]$

(a) \_\_\_\_\_

**Solution:** Use the tables:  $E[Y] = \frac{1+0}{2} = \frac{1}{2}$ .

(b) (2 points) Find  $\sigma^2 = V[Y]$

(b) \_\_\_\_\_

**Solution:** Use the tables:  $V[Y] = \frac{(1-0)^2}{12} = \frac{1}{12}$ .

(c) (5 points) Tchebysheff's Theorem gives an inequality for  $P(|Y - \mu| < 2\sigma)$ . Write this inequality in the space below:

$$P(|Y - \mu| < 2\sigma) \quad \underline{\hspace{2cm}} \quad \underline{\hspace{2cm}}$$

**Solution:**

$$P(|Y - \mu| < 2\sigma) \geq 1 - \frac{1}{2^2}$$

(d) (5 points) Find  $P(|Y - \mu| < 2\sigma)$

(d) \_\_\_\_\_

**Solution:** Note that  $\sigma = \frac{1}{2\sqrt{3}} > \frac{1}{4}$ . So the values of  $Y$  with  $|Y - \frac{1}{2}| < 2\sigma$  include the entire interval  $[0, 1]$ . Thus the probability is 1.