NO CALCULATORS Math 447 Spring 2015 NO CALCULATORS

Final Exam

May 13, 2015

- Total value 340 points. Each part valued as indicated.
- SHOW YOUR WORK unless otherwise indicated. "NO WORK" may result in "NO POINTS".
- Simplify your answers when possible. Do the arithmetic, remove parentheses, reduce fractions, etc.
- Cross out anything you don't want graded!
- Use the back sides of pages if you need extra space. If you have anything on a back side that you want graded, indicate where it is.

Student: ______________________________

Section (please circle): 01

Problem #	Possible Points	Points	Problem #	Possible Points	Points
I	24		VIII	40	
II	24		IX	25	
III	37		X	32	
IV	24		XI	30	
V	24		XII	24	
VI	18		XIII	20	
VII	18				
Total				340	

I. (24 points. 6 points each.) $\{A, B, C\}$ is an independent collection of events with $P(A) = \frac{1}{3}$, $P(B) = \frac{1}{4}$, and $P(C) = \frac{1}{5}$.

(1) $P\{\text{exactly two of the events } A,\ B,\ C \text{ occurs}\} = ?$

(2) $P(B \mid A \cup B) = ?$

(3) $P\{A \cup B \mid B \cap C\} = ?$

(4) Now suppose $D \cap A = \emptyset$ and $P(D) = \frac{1}{5}$.
$P\{A \cap B \cap C \cap D\} = ?$

II. (24 points. Each part valued as marked.) $\{X_1, X_2, \cdots\}$ is an independent sequence of Poisson random variables with a mean λ, that is

$$P(X_k = x) = \frac{\lambda^x e^{-\lambda}}{x!}, \quad x = 0, 1, 2, \ldots$$

for all k. Define a new random variable $S_n = \sum_{k=1}^{n} X_k$.

(1) (7 points) Give the probability mass function (PMF) of $S_2 = X_1 + X_2$. I.e., $P\{S_2 = x\} = ?$ for $x = 0, 1, 2, \ldots$.

(2) (4 points) $E(S_{100}) = E\left(\sum_{k=1}^{100} X_k\right) =?$

(3) (4 points) $\mathrm{Var}(S_{100}) = ?$

(4) (4 points) What distribution does S_{100} have? Give $P\{S_{100} = x\} =?$ for $x = 0, 1, 2, \ldots$.

(5) (5 points) $P\{X_1 = 4 \mid X_1 + X_2 + X_3 = 10\} = ?$ (Leave your answer as a function of λ)

III. (37 points. Each part valued as marked) The voters in a small town consist of 5000 democrats, 4000 republicans, and 1000 independents. A "random sample" of 50 voters is taken. You may leave your answers in terms of binomial/multinomial coefficients or combinations/permutations.

(1) (5pt) Find the probability that the sample consists of 25 democrats, 20 republicans, and 5 independents.

(2) (5pt) Find the probability that there are exactly 15 republicans among the 50 voters.

(3) (12 points) Give both the binomial and poisson approximations to your answer in (2). Put your answers on the back of page 3.

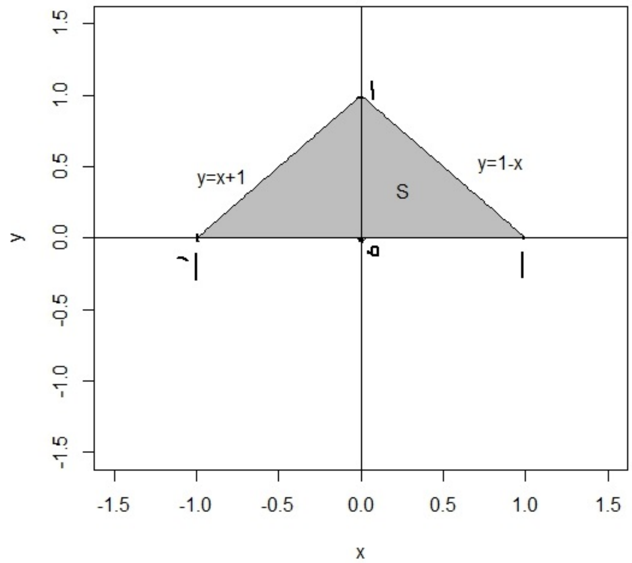
Now suppose these 50 voters were taken in order,

(4) (5 pts) what is the probability that the first 6 voters taken consists of 2 democrats, 2 republicans, and 2 independent? (not necessarily in that order)

(5) (5 pts) What is the probability that none of the last 30 voters taken are republicans.

(6) (5 pts) what is the probability that the 30th voter taken is a republican? Put your answer on the back of page 3.

IV. (24 points.) A point (X, Y) is chosen "at random" (equal areas are equally likely) from the sample space $\{(x, y) \mid |x| + |y| \leq 1, y \geq 0\}$ (the shaded area). Define random variables $X(x, y) = x$ and $Y(x, y) = y$.



(1) (6 pts) $P\{Y \leq \frac{1}{2}\} = ?$

(2) (6 pts) $P\{X^2 + Y^2 \leq 1\} = ?$

(3) (4 pts) $P\{X \leq \frac{1}{2}\} = ?$

(4) (8 pts) Give the distribution function of the random variable X.
I.e., $F_X(x) = P\{X \leq x\} \;= ?$ for all x.

V. (24 points. 12 points each.) Tom is selling a product in an area where 30% of the people live in the city; the rest live in the suburbs. Tom knows that 20% of the city residents (urbanites) use this product; and 10% of the suburb residents use this product.

1) What fraction of the people in the area (regardless of living in the city or the suburbs) use this product?

2) Given that a person in this area is **NOT** using this product, what is the probability that he lives in the city?

VI. (18 points.) X has probability density function (PDF)

$$f_X(x) = \begin{cases} \frac{1}{4} & -1 < x < 3 \\ 0 & \text{otherwise.} \end{cases}$$

Let $Y = X^2$. Find the probability <u>density</u> function (PDF) of Y. I.e., $f_Y(y) = ?$ (Note that the range of y where $f_Y(y) > 0$ is important here.)

VII. (18 points. 6 points each) Suppose that the random variable X in has a moment generating function $M_X(t) = \frac{1}{1-t^2}$ for $|t| < 1$. (you are not supposed to know the distribution of X)

(1) $EX^2 = ?$

(2) $EX^{20} = ?$

(3) Let $Y = 3X + 4$, find the moment generating function of Y, i.e. $M_Y(t) = Ee^{tY} =?$

VIII. (40 points. Each part valued as indicated.) X has distribution function (CDF)

$$F(x) = \begin{cases} 0, & x < 0, \\ (1+x)/8, & 0 \le x < 3, \\ x^2/16, & 3 \le x < 4, \\ 1, & x \ge 4. \end{cases}$$

(1) (4 points.) $P\{X \le 1\} = ?$

(2) (4 points.) $P\{3 < X < 4\} =?$

(3) (8 points.) $EX = ?$ (Leave your answer as integrals with correct bounds)

(4) (8 points.) Let $Y = e^X$, find $E(Y) =?$ (Leave your answer as integrals with bounds)

Now let $Y = \frac{1}{2}|X|$.

(5) (6 points) $P\{X \ge 1 \mid Y < \frac{3}{2}\} = ?$

(6) (10 points) $F_Y(y) = P\{Y \le y\} = ?$ For all y. Put your answer and work on the back of page 7.

IX. (25 points) X and Y have joint density

$$f(x,y) = \begin{cases} 2e^{-(x+y)}, & 0 < x < y < \infty \\ 0, & \text{otherwise.} \end{cases}$$

Let the random variables U and V defined as

$$U = e^{2X}$$
$$V = Y - X.$$

Note that the definition for $0 < X < Y < \infty$ is irrelevant.

(1) (5 points.) Find $P(Y < 2) =$? (Leave your answer as an integral. Make sure your limits of integration are correct.)

(2) (15 points.) Find the joint probability density function for U and V. I.e.,

$$f_{U,V}(u,v) = g(u,v) = ?$$

Be sure to specify where $g(u,v) > 0$!

(3) (5 points.) Are random variables U and V independent? (Give your arguments, no argument, no point.)

X. (32 points. Each part valued as indicated.)

$$X \text{ and } Y \text{ have joint density } f(x,y) = \begin{cases} 2e^{-(x+y)} & 0 < x < y < \infty \\ 0 & \text{otherwise.} \end{cases}$$

(1) (5 points.) The marginal density function of Y, i.e. $f_Y(y) =$? (Specify the range!)

(2) (5 points.) The marginal density function of X, i.e. $f_X(x) =$? (Specify the range!)

(3) (10 points.) $f_{Y|X}(y \mid x) =$? (Specify the range!)

(4) (6points.) $E\{Y \mid X = x\} =$? (Specify the range!)

(5) (6 points.) $U = e^{-2Y}$. $E\{U \mid X = x\} =$? (Specify the range!)

XI. (30 points.) Suppose $Y_1, Y_2, \ldots, Y_n$ independently follow a uniform distribution on the interval $[0, \theta]$ with a density function

$$f(y) = \begin{cases} \frac{1}{\theta}, & 0 \le y \le \theta; \\ 0, & elsewhere, \end{cases}$$

where $\theta > 0$.

(1) (4 pts) Find CDF of Y_1, i.e. $F_{Y_1}(y) =?$ (Not the first order statistic $Y_{(1)}$!) (Specify the range!)

(2) (8 pts) Define $Y_{(n)} = \max(Y_1, Y_2, \ldots, Y_n)$. Find the probability density function (PDF) of $Y_{(n)}$; (Specify the range!)

(3) (8 pts) Find $E[Y_{(n)}^2] =?$

(4) (5 pts) Find $P(Y_1 < Y_2 < Y_3 < Y_4) =?$

(5) (5 pts) Let $Y_{(k)}$ be the kth order statistic from the "sample" $Y_1, \ldots, Y_n$ and $n = 12$. Find $P(Y_6 < Y_{(6)}$ or $Y_6 > Y_{(10)}) =?$

XII. (24 points. Each part valued as indicated.) According to the National Center for Health Statistics, the distribution of serum cholesterol levels for 20- to 74-year-old males living in the United States has a mean 211 mg/dl, and a standard deviation of 50 mg/dl.

(1) (12 pts) We are planning to collect a sample of 100 individuals and measure their cholesterol levels. What is the probability that our sample mean will be above 221 mg/dl? (use the attached SOA table)

(2) (12 pts) How large should the sample size n be in order to ensure a 90% probability that the sample mean will be within 5 mg/dl of the true mean? (use the attached SOA table)

XIII. (20 points. Each part valued as indicated.) Suppose $X_1, X_2, \cdots$ is an i.i.d. sequence of **Poisson** random variables with a mean λ.

(1) (5 points.) $E\left[\frac{1}{n}\sum_{k=1}^{n} X_k\right] = ?$ (Put your answer as a function of λ, otherwise no point)

(2) (5 points.) $E\left[\frac{1}{n}\sum_{k=1}^{n} X_k^2\right] = ?$ (Put your answer as a function of λ, otherwise no point)

(3) (5 points.) Does the sequence $\left\{\frac{1}{n}\sum_{k=1}^{n} X_k^2 + \sqrt{\frac{1}{n}\sum_{k=1}^{n} X_k},\ n = 1, 2, \cdots\right\}$ have a limit (in any sense)? If so, what is it, and in what sense? (Put your answer as a function of λ, otherwise no point)

(4) (5 points.) Give a sequence of random variables, $Y_n = h_n(X_1, \ldots, X_n)$, which are functions of the X_k's, such that $Y_n \xrightarrow{\text{a.s.}} e^{\lambda^2}$.

Standard Normal Probabilities

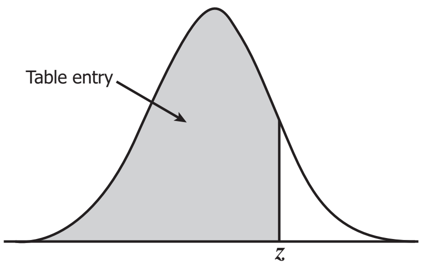


Table entry for z is the area under the standard normal curve to the left of z.

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998