Math 447 Spring 2015
Exam 3
April 21, 2015

- Total value 220 points. Each part valued as indicated.
- SHOW YOUR WORK unless otherwise indicated. "NO WORK" may result in "NO POINTS".
- Simplify your answers when possible. Do the arithmetic, remove parentheses, reduce fraction, etc.
- Cross out anything you don't want graded!
- Use the back sides of pages if you need extra space. If you have anything on a back side that you want graded, indicate where it is.

Student: Solution

Section (please circle): 01 (Xu)

Problem #	Possible Points	Points
I	30	
II	14	
III	21	
IV	36	
V	30	
VI	28	
VII	28	
VIII	33	
Total	220	

I. (30 points.) X has probability mass function (PMF; i.e., $p(x) = P\{X = x\}$) given in the table below

x	-2	0	2	otherwise
$p(x) = P\{X = x\}$	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{1}{6}$	0

(1) (8 points) Give the distribution function (CDF) of X.
I.e., $F(x) = P\{X \leq x\} = ?$

$$F(x) = \begin{cases} 0, & x < -2 \\ \frac{1}{6}, & -2 \leq x < 0 \\ \frac{5}{6} & 0 \leq x < 2 \\ 1 & x \geq 2. \end{cases}$$

(2) (6 points) $EX^4 = ?$

$$= (-2)^4 \times \frac{1}{6} + 0^4 \times \frac{2}{3} + 2^4 \times \frac{1}{6} = \frac{16}{6} + \frac{16}{6} = \frac{32}{6} = \frac{16}{3}$$

(3) (6 points) Var $X = ?$

$$EX = (-2) \times \frac{1}{6} + 0 \times \frac{2}{3} + 2 \times \frac{1}{6} = 0$$

$$EX^2 = (-2)^2 \times \frac{1}{6} + 0^2 \times \frac{2}{3} + 2^2 \times \frac{1}{6} = \frac{4}{3} \quad \Rightarrow Var(X) = EX^2 - (EX)^2 = \frac{4}{3} - 0^2$$

$$= \frac{4}{3}.$$

(4) (6 points) Define $Y = \frac{1}{X+1}$. Give the moment generating function (MGF) of Y, $m_Y(t) = ?$.

$$M_Y(t) = Ee^{tY} = Ee^{\frac{t}{X+1}} = e^{\frac{t}{-2+1}} P(X=-2) + e^{\frac{t}{0+1}} P(X=0)$$

$$+ e^{\frac{t}{2+1}} P(X=2)$$

$$= \frac{1}{6}e^{-t} + \frac{2}{3}e^{t} + \frac{1}{6}e^{\frac{t}{3}}$$

(5) (4 points)Find the qth quantile of X with $q = \frac{1}{4}$. (No work need be shown.)

$\boxed{0}$

II. (14 points.) $\{X_1, \ldots, X_n\}$ is an independent collection of random variables having finite mean μ and finite variance σ^2. Let $S = \sum_{k=1}^{3} kX_k$. Your answers to the questions below will involve μ and σ.

(1) (4 points.) $E(S) = ?$

$$E(S)=E\left(\sum_{k=1}^{3} kX_k\right)=\sum_{k=1}^{3} kE(X_k)=\sum_{k=1}^{3} k\mu=\mu(1+2+3)$$
$$=6\mu$$

(2) (4 points) $Var(S) = ?$

(independent)

$$Var(S)=Var\left(\sum_{k=1}^{3} kX_k\right)=\sum_{k=1}^{3} Var(kX_k)\, + 2\sum_{k=1}^{3}\sum_{j=2}^{3} cov(kX_k, jX_j)$$
$$=\sum_{k=1}^{3} k^2 Var(X_k) - 0 = \sum_{k=1}^{3} k^2 \cdot \sigma^2 = (1+2^2+3^2)\sigma^2 = 14\sigma^2$$

(3) (6 points) If $Y = X_1 + 2X_2 - 3$ and $Z = X_1 - 2X_2 + X_3 + 4$, then $\mathrm{Cov}(Y, Z) = ?$

$$cov(Y,Z) = cov(X_1+2X_2\cancel{-3}, X_1-2X_2+X_3\cancel{+4})$$
$$=cov(X_1+2X_2, X_1-2X_2+X_3)$$
$$=cov(X_1,X_1) - 2\,cov\cancel{(X_1,X_2)} + cov\cancel{(X_1,X_3)}$$
$$+(2)\,cov\cancel{(X_2,X_1)} + (2)(-2)\,cov(X_2,X_2)$$
$$+2\,cov\cancel{(X_2,X_3)}$$
$$=Var(X_1) - 4\,Var(X_2)$$
$$=\sigma^2 - 4\sigma^2$$
$$=-3\sigma^2$$

III. (21 points.) X has probability density function (PDF)

$$f_X(x) = \begin{cases} \dfrac{3x^2}{8} & 0 < x < 2 \\ 0 & \text{otherwise.} \end{cases}$$

(1) (6 points) Give the distribution function (CDF) of X.
I.e., $F(x) = P\{X \le x\} = ?$ (Be sure to give all pieces of this function. Otherwise you will lose points.)

$$F(x) = \int_{-\infty}^{x} f_X(t)\,dt = \begin{cases} 0, & x \le 0 \\ \int_0^x \frac{3}{8}t^2\,dt, & 0<x<2 \\ 1, & x \ge 2 \end{cases} = \begin{cases} 0, & x \le 0 \\ \frac{1}{8}x^3, & 0<x<2 \\ 1, & x \ge 2 \end{cases}$$

(2) (6 points) $EX = ?$ $\int_{-\infty}^{+\infty} x f_X(x)\,dx = \int_0^2 \frac{3}{8}x^3\,dx = \frac{3}{32}x^4\Big|_0^2$

$= \frac{3}{32} \times 16 = \frac{3}{2}$

(3) (3 points) $P\{X = 1/3\} = ?$ 0 since X is continuous.

(4) (6 points) $P\{X^2 < \frac{1}{4}\} = ?$ $P\left(-\frac{1}{2} < X < \frac{1}{2}\right) = P\left(0 < x < \frac{1}{2}\right)$

$= F\left(\frac{1}{2}\right) - F(0) = \frac{1}{8} \times \left(\frac{1}{2}\right)^3 = \frac{1}{64}$

IV. (36 points.) X has distribution function (CDF)

$$F_X(x) = \begin{cases} 0 & x < 0 \\ \dfrac{1+x^2}{10} & 0 \le x < 2 \\ 1 & 2 \le x \end{cases}$$

(1) (5 points) Does this distribution function (CDF) have any jumps? If yes, identify all the jump points and the associated probabilities (i.e. $P(X = x) = ?$).

$$P(X=0) = F(0) - F(0-) = \frac{1+0^2}{10} - 0 = \frac{1}{10}$$

$$P(X=2) = F(2) - F(2-) = 1 - \frac{1+2^2}{10} = \frac{1}{2}$$

(2) (5 points) $P\left\{-\pi < X \le \frac{1}{2}\right\} = ?$ $F(\frac{1}{2}) - F(-\pi) = \frac{1+(\frac{1}{2})^2}{10} - 0 = \frac{1}{8}$

(3) (5 points) $P\{0 \le x \le 2\} = ?$ $F(2) - F(0-) = 1 - 0 = 1$

(4) (3 points) A is the set $\left\{-1, 0, \frac{1}{\sqrt{2}}, 2\right\}$. $P\{X \notin A\} = ?$

$$= 1 - P(X \in A) = 1 - \left(P(X=0) + P(X=2) + 0 + 0\right)$$

$$= 1 - \left(\frac{1}{10} + \frac{1}{2}\right) = \frac{2}{5}$$

(5) (2 points) Find $F'(x) =?$ $\begin{cases} \frac{x}{5}, & 0 < x < 2 \\ 0, & \text{elsewhere.} \end{cases}$

(6) (8 points) $EX = ?$ $\int_{-\infty}^{+\infty} x F'(x)\,dx + 0 \times p(X=0) + 2p(X=2)$

$$= \int_0^2 \frac{x^2}{5}\,dx + 2 \times \frac{1}{2}$$

$$= \frac{x^3}{15}\Big|_0^2 + 1 = \frac{23}{15}$$

(7) (8 points) $Var(X) = ?$ (give a summation of reduced fractions as your final answer, no need to simplify.)

$$EX^2 = \int_{-\infty}^{+\infty} x^2 F'(x)\,dx + 0^2 p(X=0) + 2^2 p(X=2).$$

$$= \int_0^2 \frac{x^3}{5}\,dx + 4 \times \frac{1}{2} = \frac{1}{20} x^4 \Big|_0^2 + 2 = \frac{14}{5}$$

$$Var(X) = EX^2 - (EX)^2 = \frac{14}{5} - \left(\frac{23}{15}\right)^2 = \frac{101}{225}.$$

V. (30 points.) X is normally distributed with mean $\mu = 3$ and variance $\sigma^2 = 36$.

(1) (5 points) $P\{3 \le X \le 15\} = ?$ $P(\frac{3-3}{6} \le Z \le \frac{15-3}{6}) = P(0 \le Z \le 2)$

$= \Phi(2) - \Phi(0) = 0.9772 - 0.5 = 0.4772$

(2) (5 points) $P\{|X-6| \le 9\} = ?$ $P(-3 \le X \le 15) = P(\frac{-3-3}{6} \le Z \le \frac{15-3}{6})$

$= P(-1 \le Z \le 2) = \Phi(2) - \Phi(-1) = \Phi(2) - (1 - \Phi(1))$

$= \Phi(2) + \Phi(1) - 1 = 0.9772 + 0.8413 - 1 = 0.8185$

(3) (5 points) Find out a value for x_0 so that $P\{X \ge x_0\} = 0.99$

$P(X \ge x_0) = 0.99 \Rightarrow P(Z \ge \frac{x_0 - 3}{6}) = 0.99$

$\Rightarrow P(Z \le -\frac{x_0-3}{6}) = 0.99 \Rightarrow -\frac{x_0-3}{6} = Z_{0.99} = 2.33$

$\Rightarrow x_0 = 3 - 2.33 \times 6 = \boxed{-10.98}$

(4) (5 points) $P\{X^3 + 27 \ge 0\} = ?$

$= P(X^3 \ge -27) = P(X \ge -3) = P(Z \ge \frac{-3-3}{6}) = P(Z \ge -1)$

$= P(Z \le 1) = \Phi(1) = 0.8413$

(5) (5 points) Give the moment generating function (MGF) for X, $m_X(t) = ?$

$M_X = e^{3t + \frac{36}{2}t^2} = e^{3t + 18t^2}$

(6) (5 points) Let $Y = 5X + 10$. Give the probability *density* function (PDF) for Y. I.e., $f_Y(x) = ?$

$EY = 5EX + 10 = 5 \times 3 + 10 = 25$

$Var(Y) = Var(5X + 10) = 5^2 Var(X) = 5^2 \cdot 6^2 = 30^2$

$Y \sim N(25, 30^2)$ $f_Y(y) = \frac{1}{\sqrt{2\pi} \cdot 30} e^{-\frac{(y-25)^2}{2 \times 30^2}}$ $-\infty < y < +\infty$

VI. (28 points.) X has a *cumulative distribution function* (CDF). (If you don't pay attention to the difference between CDF and PDF, you can say goodbye to the following 33 pts)

$$F_X(x) = \begin{cases} 1 - e^{-5x} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

$X \sim exp(\frac{1}{5})$.

$\Rightarrow EX = \frac{1}{5}, \ Var(X) = \frac{1}{5^2}$

(1) (4 points) $E(5X + 1) = ?$

$= 5EX + 1 = 5 \times \frac{1}{5} + 1 = 2$

(2) (4 points) $\text{Var}(5X + 1) = ?$

$Var(5X+1) = 5^2 \, Var(X) = 5^2 \times \frac{1}{5^2} = 1$

(3) (5 points) Give the moment generating function (MGF) of X, $m_X(t) = ?$

$M_X(t) = \dfrac{1}{1 - \frac{1}{5}t}$ for $(t < 5)$.

(4) (5 points) $E[(5X)^{20}] = ?$

$E[(5X)^{20}] = 5^{20} EX^{20} = 5^{20} \cdot 20! \left(\frac{1}{5}\right)^{20} = 20!$

(5) (5 points) $P\{X \leq \frac{1}{10} \mid X \geq \frac{1}{20}\} = ?$

$= \dfrac{P(X \leq \frac{1}{10}, X \geq \frac{1}{20})}{P(X \geq \frac{1}{20})} = \dfrac{P(\frac{1}{20} \leq X \leq \frac{1}{10})}{P(X \geq \frac{1}{20})} = \dfrac{F_X(\frac{1}{10}) - F_X(\frac{1}{20})}{1 - F_X(\frac{1}{20})}$.

$= \dfrac{(1 - e^{-5 \times \frac{1}{10}}) - (1 - e^{-5 \times \frac{1}{20}})}{1 - (1 - e^{-5 \times \frac{1}{20}})} = \dfrac{e^{-\frac{1}{4}} - e^{-\frac{1}{2}}}{e^{-\frac{1}{4}}} = \boxed{1 - e^{-\frac{1}{4}}}$

(6) (5 points) Let $Y = 10X$. Give the distribution function (CDF) of Y. I.e., $F_Y(y) =$?
Moreover, based on $F_Y(y)$, tell what distribution Y has (and what is/are the parameter(s)).

$F_Y(y) = P(Y \leq y) = P(10X \leq y) = P(X \leq \frac{y}{10}) = F_X(\frac{y}{10})$.

$$= \begin{cases} 1 - e^{-5 \times \frac{y}{10}}, & \frac{y}{10} > 0 \\ 0, & \frac{y}{10} \leq 0 \end{cases} = \begin{cases} 1 - e^{-\frac{y}{2}}, & y > 0 \\ 0, & y \leq 0 \end{cases}$$

Hence, $Y \sim$ exponential (β) with $\beta = 2$.

VII. (28 points.) X and Y have joint probability mass function given in the table below.

$y \backslash x$	1	2	otherwise
0	$\frac{1}{10}$	$\frac{2}{10}$	0
1	$\frac{3}{10}$	$\frac{4}{10}$	0
otherwise	0	0	0

(1) (5 points) $E(XY) = ?$

$$E(XY) = (1\times 0)\cancel{p(1,0)} + (1\times 1)p(1,1) + (2\times 0)\cancel{p(2,0)} + (2\times 1)p(2,1)$$

$$= 1\times\frac{3}{10} + 2\times\frac{4}{10} = \frac{11}{10}$$

(2) (5 points) $Cov(X, Y) = ?$ $E(XY) - (EX)(EY) = \frac{11}{10} - \frac{8}{5}\times\frac{7}{10} = \boxed{-\frac{1}{50}}$

$$EX = 1\times\left(\frac{1}{10}+\frac{3}{10}\right) + 2\times\left(\frac{2}{10}+\frac{4}{10}\right) = \frac{8}{5}$$

$$EY = 0\times\left(\frac{1}{10}+\frac{2}{10}\right) + 1\times\left(\frac{3}{10}+\frac{4}{10}\right) = \frac{7}{10}$$

(3) (5 points) Are X and Y independent of each other? (give your arguments. no argument, no point.)

Solution I: $P(X=1, Y=0) = \frac{1}{10} \neq P(X=1)P(Y=0) = \frac{4}{10}\times\frac{3}{10} = \frac{3}{25}$

Solution II: $cov(X,Y) = -\frac{1}{50} \neq 0$

$\Rightarrow$ X and Y are NOT independent

(4) (8 points) Give $P\{Y = y \mid X = x\}$ for all values of x and y. (You need to organize them as a "table".)

$Y \backslash X$	1	2
0	$\frac{1}{4}$	$\frac{1}{3}$
1	$\frac{3}{4}$	$\frac{2}{3}$

(5) (5 points) $E\{Y \mid X = 1\} = ?$

$$E(Y|X=1) = 0\times P(Y=0|X=1) + 1\times P(Y=1|X=1)$$

$$= 0 + 1\times\frac{3}{4} = \frac{3}{4}$$

VIII. (33 points.) X and Y have joint probability *density* function (PDF)

$$f(x,y) = \begin{cases} \frac{3}{4}y & 0 < x < 2,\ 0 < y < 2,\ x + y < 2 \\ 0 & \text{otherwise.} \end{cases}$$

(1) (5 points) Give the (marginal) probability density (marginal PDF) $f_X(x)$ of X. *(Be sure to specify the regions of x.)*

$$f_X(x) = \int_{-\infty}^{+\infty} f(x,y)\,dy = \begin{cases} \int_0^{2-x} \frac{3}{4}y\,dy = \frac{3}{8}(2-x)^2, & x \in (0,2) \\ 0, & x \notin (0,2) \end{cases} = \begin{cases} \frac{3}{8}(2-x)^2, & 0 < x < 2 \\ 0, & x \notin (0,2). \end{cases}$$

(2) (5 points) Give the (marginal) probability density (marginal PDF) $f_Y(y)$ of Y. *(Be sure to specify the regions of y.)*

$$f_Y(y) = \int_{-\infty}^{+\infty} f(x,y)\,dx = \begin{cases} \int_0^{2-y} \frac{3}{4}y\,dx = \frac{3}{4}y \cdot x\Big|_0^{2-y} = \frac{3}{4}y(2-y), & 0 < y < 2 \\ 0, & y \notin (0,2) \end{cases}$$

(3) (4 points) Are X and Y independent of each other? (give your arguments. no argument, no points!)

No, X and Y are NOT independent

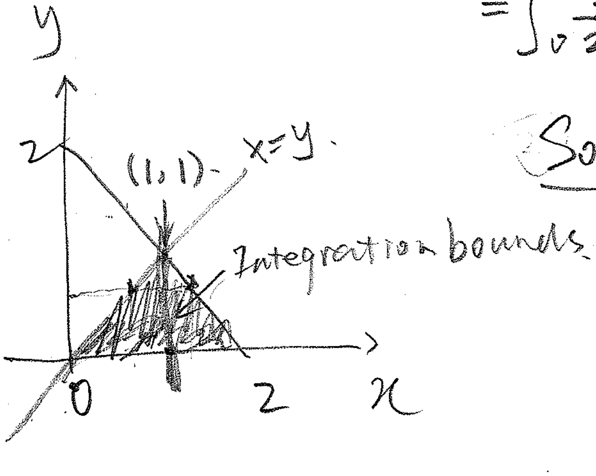
because $f(x,y) \neq f_X(x)\, f_Y(y)$.

(4) (5 points) $P(X > Y) = ?$ (Leave your answer as an integral. Make sure your limits of integration are correct.)

(dx) first and then (dy).

Solution ① $P(X>Y) = \int_0^1 \left(\int_y^{2-y} \frac{3}{4}y\,dx \right) dy = \int_0^1 \frac{3}{4}y(2-2y)\,dy$

$$= \int_0^1 \frac{3}{2}(y - y^2)\,dy = \left(\frac{3}{4}y^2 - \frac{1}{2}y^3 \right)\Big|_0^1 = \frac{3}{4} - \frac{1}{2} = \boxed{\frac{1}{4}}$$



Solution ② ←(dy) first and then (dx)

$$P(X>Y) = \int_0^1 \left(\int_0^x \frac{3}{4}y\,dy \right) dx + \int_1^2 \left(\int_0^{2-x} \frac{3}{4}y\,dy \right) dx$$

$$= \int_0^1 \frac{3}{8}x^2\,dx + \int_1^2 \frac{3}{8}(2-x)^2\,dx$$

$$= \frac{1}{8}x^3\Big|_0^1 + \frac{1}{8}(x-2)^3\Big|_1^2 = \boxed{\frac{1}{4}}$$

(5) (6 points) Give the conditional probability density function (conditional PDF) of Y given that $X = x$. I.e., $f_{Y|X}(y \mid x) = ?$ *(Be sure to specify the region of (x, y) in which your result holds.)*

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \begin{cases} \left.\begin{array}{ll} \frac{\frac{3}{4}y}{\frac{3}{8}(2-x)^2} = \frac{2y}{(2-x)^2}, & 0<y<2-x \\ 0, & y \notin (0, 2-x) \end{array}\right\} x \in (0,2) \\ \text{undefined}, \quad x \notin (0,2) \end{cases}$$

(6) (4 points) $E\left\{Y \mid X = \frac{1}{2}\right\} = ?$

$$E(Y|X=\tfrac{1}{2}) = \int_{-\infty}^{+\infty} y f_{Y|X=\frac{1}{2}}(y)\,dy = \int_0^{2-\frac{1}{2}} \frac{2y^2}{(2-\frac{1}{2})^2}\,dy$$

$$= \int_0^{\frac{3}{2}} \frac{8}{9} y^2\,dy = \frac{8}{27} y^3 \Big|_0^{\frac{3}{2}} = \frac{8}{27} \times \frac{27}{8} = 1$$

(7) (4 points) $E\left\{(Y + 1)X \mid X = \frac{1}{2}\right\} = ?$

$$E((Y+1)X|X=\tfrac{1}{2}) = E\left(\frac{Y+1}{2}\Big|X=\tfrac{1}{2}\right)$$

$$= \frac{1}{2}E(Y|X=\tfrac{1}{2}) + \frac{1}{2}$$

$$= \frac{1}{2} \times 1 + \frac{1}{2}$$

$$= 1$$

Standard Normal Probabilities

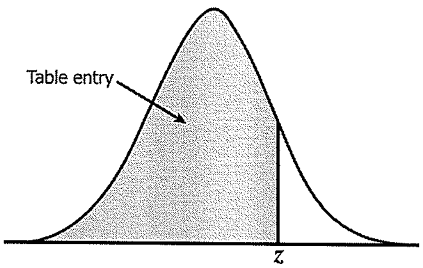


Table entry for z is the area under the standard normal curve to the left of z.

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998