NO CALCULATORS Math 447 Fall 2014 NO CALCULATORS

Final Exam

December 15, 2014

- Total value 340 points. Each part valued as indicated.
- SHOW YOUR WORK unless otherwise indicated. "NO WORK" may result in "NO POINTS".
- Simplify your answers when possible. Do the arithmetic, remove parentheses, reduce fractions, etc.
- Cross out anything you don't want graded!
- Use the back sides of pages if you need extra space. If you have anything on a back side that you want graded, indicate where it is.

Student: ______________________________

Section (please circle): 01 02

Problem #	Possible Points	Points	Problem #	Possible Points	Points
I	24		VIII	15	
II	30		IX	49	
III	36		X	33	
IV	24		XI	20	
V	24		XII	35	
VI	15		XIII	16	
VII	40				
Total				340	

I. (24 points. 6 points each.) $\{A, B, C\}$ is an independent collection of events with $P(A) = \dfrac{1}{3}$, $P(B) = \dfrac{1}{4}$, and $P(C) = \dfrac{1}{5}$; $D \cap A = D \cap B = 0 \cap C \neq \emptyset$; and $P(D) = \dfrac{1}{5}$.

(1) $P\{\text{exactly one of the events } A,\ B,\ C \text{ occur}\} = ?$

(2) $P(B \cup C \mid A) = ?$

(3) $P\{A \cap B \mid B \cup C\} = ?$

(4) $P\{A \cup B \cup C \mid D^c\} = ?$

II. (30 points. Each part valued as marked.) $\{X_1, X_2, \cdots\}$ is an independent sequence of Bernoulli random variables with $P\{X_k = 1\} = p$ $(0 < p < 1)$ and

$P\{X_k = 0\} = q = 1 - p$ for all k. $S_n = \sum_{k=1}^{n} X_k$

(1) (points) Give the distribution function of S_2. I.e., $P\{S_2 \leq x\} = ?$ for all x.

(2) (points) $P\{S_{100} = k\} =?$ for $k = 0, 1, \ldots, n$.

(3) (points) $ES_{100} =?$

(4) (points) Var $S_{100} = ?$

(5) (points) $P\{S_{100} = 30 \mid S_{50} = 10\} = ?$

(6) (points) $P\{X_1 + X_3 = 1 \mid S_2 = 1\} = ?$

III. (points. Each part valued as marked) A lottery sells 100,000 tickets. 100 are winning tickets of a major prize, 900 are winning tickets of a minor prize, and the other 99,000 are non-winning tickets. A speculator buys 200 tickets.

(1) (points) What is the probability that exactly 20 of his tickets are winning tickets?

(2) (points) What is the probability that exactly 15 of his tickets are winners of a minor prize and exactly 5 are winners of a major prize?

(3) (points) Give both the binomial and poisson approximations to your answer in (1).

Now suppose the tickets were purchased one at a time, in order (i.e., 1st ticket purchased, 2nd ticket purchased, ...).

(4) (points) What is the probability that none of the 200 tickets is a winning ticket?

(5) (points) What is the probability that none of the first 50 tickets purchased are winners and none of the last 50 tickets purchased are winners?

(6) (points) What is the probability that, of the 200 tickets purchased, there is the same number of tickets for a major prize as there is for a minor prize?

IV. (27 points. 6 points each.) A point (X, Y) is chosen "at random" (equal areas are equally likely) from the triangle $\{(x, y) \mid 0 \leq x < 4,\ 0 < y < x\}$. $X(x, y) = x$ and $Y(x, y) = y$.

(1) $P\{X \leq 1\} = ?$

(2) $P\{Y \leq X^2\} = ?$

(3) Give the distribution function of the random variable X.
I.e., $F_X(x) = P\{X \leq x\} \ = ?$ for all x.

(4) $P\{X \leq 1 \mid Y \leq 2\} = ?$

V. (24 points. 12 points each.) A nickel with $P\{\text{heads}\} = \frac{1}{3}$ is tossed. If the nickel comes up heads, a quarter with $P\{\text{heads}\} = \frac{1}{4}$ is tossed. If the nickel comes up tails, a "fair" quarter (i.e. $P(\text{heads}) = \frac{1}{2}$) is tossed.

1) What is the probability that the result of the second (i.e., quarter) toss is heads?

2) Given that the result of tossing the quarter is <u>tails</u>, what is the probability that the result of the "nickel toss" was tails?

VI. (15 points.) X has probability density function (PDF)

$$f_X(x) = \begin{cases} \dfrac{3x^2}{8} & 0 < x < 2 \\ 0 & \text{otherwise.} \end{cases}$$

$Y = e^X$. Find the probability density function (PDF) of Y. I.e., $f_Y(y) = ?$

VII. (40 points. Each part valued as indicated.) X has distribution function (CDF)

$$F(x) = \begin{cases} 0 & x < 0 \\ \dfrac{1 + x^2}{20} & 0 \leq x < 4 \\ 1 & 2 \leq x \end{cases}$$

(1) (4 points.) $P\{X < 1\} = ?$

(2) (4 points.) $P\{-\pi < X < 20\} =?$

(3) (12 points.) $EX = ?$

(4) (4 points.) $P\{X \text{ is irrational}\} = ?$

Now let $Y = X^2$.

(5) (points) $P\{X \geq 1 \mid Y \leq 4\} = ?$

(6) (points) $F_Y(y) = P\{Y \leq y\} = ?$ For all y.

VIII. (18 points. 6 points each) X in $N(0, \sigma^2)$ $\sigma > 0$.

(1) $\mathrm{MGF}_X(t) = Ee^{tX} = ?$

(2) $EX^4 = ?$

(3) $E(X-1)^4 = ?$

IX. (49 points. Each part valued as indicated.)

X and Y have joint density $f(x,y) = \begin{cases} Cxy & 0 < y < x < 1 \\ 0 & \text{otherwise.} \end{cases}$

(1) (5 points.) $C = ?$

(2) (5 points.) $f_X(x) = ?$

(3) (6 points.) $f_{Y|X}(y \mid x) = ?$

(4) (6points.) $E\{Y \mid X = x\} = ?$

(5) (6 points.) $E\{Y^2 \mid X = x\} = ?$

IX. – CONTINUED –

(6) Let $U = X$ and $W = \dfrac{Y}{X}$.

(a) (15 points.) Find the joint density (joint PDF) of U and W. I.e., $g(u, w) = g_{U,W}(u, w) =$? Be sure to specify the set on which the joint density is positive.

(b) (6 points.) $E(W) =$?

X. (33 points. Each part valued as indicated.) Suppose $X_1, X_2, \cdots$ is an i.i.d. sequence of random variables with finite mean μ and finite variance σ^2.

(1) (5 points.) $E\left[\frac{1}{n}\sum_{k=1}^{n} X_k\right] = ?$

(2) (5 points.) $\text{Var}\left[\frac{1}{n}\sum_{k=1}^{n} X_k\right] = ?$

(3) (5 points.) $E\left[\frac{1}{n}\sum_{k=1}^{n} X_k^2\right] = ?$

(4) (5 points.) Does the sequence $\left\{\frac{1}{n}\sum_{k=1}^{n} X_k^2(w),\ n = 1, 2, \cdots\right\}$ have a limit (in any sense)? If so, what is it, and in what sense?

(5) (5 points.) Give a sequence of random variables, $Y_n = h_n(X_1, \ldots, X_n)$, which are functions of the X_k's), such that $P\{w \mid Y_n(w) \to \sigma^2\} = 1$, i.e., $Y_n \xrightarrow{\text{a.s.}} \sigma^2$.

(6) (8 points.) Using $\Phi(t)$, the distribution function (CDF) of a standard $N(0, 1)$ random variable, give an approximation to $P\left\{\left|\frac{1}{n}\sum_{k=1}^{n} X_k - \mu\right| \le 7\right\}$.

XI. (25 points. 5 points each.) $Y, X_1, \ldots, X_n$ are independent and identically distributed (i.i.d.) observations having a continuous distribution function (CDF) $F(t)$. $X_{(k)}$ is the k-th order statistic from the "sample" $X_1, \ldots, X_n$.

(1) $P\{X_{(n)} \geq a\} = ?$

(2) If $n \geq 10$, $P\{X_{(s)} = X_7\} = ?$

(3) If F is absolutely continuous with density (PDF) $F'(x) = f(x)$, what is the density (PDF) of the distribution for $X_{(n)}$?

(4) If $n = 12$, $P\{X_{(1)} < Y < X_{(5)}\} = ?$

(5) What is the smallest n s.t. $P\{Y \in (X_{(1)}, X_{(n)}\} \geq .9$?

XII. (35 points. 7 points each.) In each problem the moment generating function $\text{mgf}_X(t) = m(t) = Ee^{tX}$ of X is given. In each problem, give each of the following:

(a) the probability density function (PDF), or probability mass function (PMF), or (cumulative) distribution function (CDF) of X – – YOUR CHOICE,

(b) EX

(c) Var X [only – (1), (2), (3) – not (4), (5)]

(d) the name of the distribution if it has one.

(1) $m(t) = \dfrac{\pi}{\pi - t} \quad t < \pi.$

(2) $m(t) = \exp\{2\pi^2 t^2 + et\}$ all t.

(3) $m(t) = \left[\dfrac{2e^t + 3}{5}\right]^n$ all t.

(4) $m(t) = \dfrac{2e^t + 2e^{-t} + 3}{7}$ all t.

(5) $m(t) = \dfrac{1}{5}\left[4\exp\left\{\dfrac{1}{2}t^2\right\} + 1\right]$ all t.