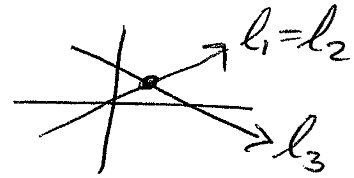
Practice Problems – Midterm Exam I

1. a) True — row operations cannot zero out a nonzero column

   b) True — $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 2 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ are row equivalent REF matrices

   c) False — RREF is unique.

   d) False — in $\mathbb{R}^2$ we can have 3 lines on 2 variables $\{x,y\}$ with one solution.



   e) True — as a matrix $\begin{array}{c} \\ m \end{array}\!\!\begin{array}{c} n \\ \begin{bmatrix} A \end{bmatrix} \end{array}$ we assume $n > m$ and as $m \geq \text{rank}(A)$ we have $n > \text{rank}(A)$.
   $\Rightarrow$ A is <u>NOT</u> 1-to-1, so A has a free variable $\Rightarrow$ ∞ or no solutions.

   f) True — homogenous SLE $[C \,\vdots\, \begin{smallmatrix} 0 \\ 0 \\ 0 \end{smallmatrix}]$ always have the trivial solution.
   if columns outnumber rows, like in (e), there are ∞ or no solutions. Since we have the trivial solution, the system has ∞ solutions

   g) True — Rank = # pivot columns = # non-zero rows in RREF

   h) True — rank 0 $\Rightarrow$ <u>no</u> pivot columns $\Rightarrow$ <u>no</u> leading entries

   i) False — they are defined to preserve rank

   j) True — from (g), the rank cannot exceed either the total # of rows <u>or</u> columns.

k) False — homogeneous systems $[C \,\vdots\, \vec{0}]$ <u>always</u> have the trivial solution.

$[C \,\vdots\, \vec{b}]$ may still have the form $\begin{bmatrix} 0 \cdots 0 & \vdots & 1 \end{bmatrix}$ and is therefore inconsistent.

l) True — if $A = [C \,\vdots\, \vec{b}]$ and $\text{rank}(A) > \text{rank}(C)$ it follows that $\text{rank}(A) = \text{rank}(C) + 1$ and the extra 1 rank is the augmented column. As the augmented column is a pivot column, A is ~~inconsistent~~ inconsistent.

m) True

n) False

o) False $(AB)^T = B^T A^T$

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2. $r_2 = R_2 - 3R_1$, $r_3 = R_3 - R_1$, $r_4 = R_4 + 2R_1$

$$\begin{bmatrix} 1 & 3 & -5 & 1 & 5 \\ 0 & 2 & -4 & 4 & -14 \\ 0 & 4 & -8 & 4 & -8 \\ 0 & 1 & -2 & 2 & -7 \end{bmatrix} \quad r_2 = R_2 - 2R_4, \; r_3 = R_3 - 4R_4$$

$$\begin{bmatrix} 1 & 3 & -5 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 20 \\ 0 & 1 & -2 & 2 & -7 \end{bmatrix} \quad \begin{matrix} r_2 = R_4 \\ r_4 = R_2 \\ r_3 = -\frac{1}{5}R_3 \end{matrix} \quad \begin{bmatrix} 1 & 3 & 1 & 5 \\ 0 & 1 & 2 & -7 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} r_1 = R_1 - R_3 \\ r_2 = R_2 - 2R_3 \end{matrix}$$

$$\begin{bmatrix} 1 & 3 & -5 & 0 & 10 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad r_1 = R_1 - 3R_2 \quad \begin{bmatrix} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

in RREF!

3. a) $\vec{x}_1, \vec{x}_2, \vec{x}_4$ are basic variables.

b) $\vec{x}_3, \vec{x}_5$ are free variables.

c) column 1, 2, 4 are pivot columns.

$$\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 2 \\ 3 \\ 1 \\ -1 \end{bmatrix} \quad \begin{bmatrix} 4 \\ 8 \\ 1 \\ -4 \end{bmatrix}$$

d) $\text{Rank}(M) = 3$

e) solutions

$$x_1 = 1 - x_5$$
$$x_2 = 2 + 2x_3 - 3x_5$$
x_3 anything
$$x_4 = -1 + x_5$$
x_5 anything

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4. Augmented Matrix

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & -1 \\ 1 & 0 & -1 & 0 \\ 6 & 2 & 1 & 1 \end{array}\right] \xrightarrow{r_1 \leftrightarrow r_2} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 2 & 1 & 1 & -1 \\ 6 & 2 & 1 & 1 \end{array}\right] \begin{array}{l} r_2 = R_2 - 2R_1 \\ r_3 = R_3 - 6R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 3 & -1 \\ 0 & 2 & 7 & 1 \end{array}\right] \xrightarrow{r_3 = R_3 - 2R_2} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right] \begin{array}{l} r_2 = R_2 - 3R_3 \\ r_1 = R_1 + R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -10 \\ 0 & 0 & 1 & 3 \end{array}\right] \longrightarrow$$ solutions equivalent to the solutions of the system

$$x_1 = 3$$
$$x_2 = -10$$
$$x_3 = 3$$

5. Augmented Matrix

$$\left[\begin{array}{cccc|c} 1 & 1 & -3 & 1 & -2 \\ 1 & 1 & 1 & -1 & 2 \\ 1 & 1 & -1 & 0 & 0 \end{array}\right] \begin{array}{l} r_2 = R_2 - R_1 \\ r_3 = R_3 - R_1 \end{array} \left[\begin{array}{cccc|c} 1 & 1 & 3 & 1 & -2 \\ 0 & 0 & -2 & -2 & 4 \\ 0 & 0 & -4 & -1 & 2 \end{array}\right] r_2 = \tfrac{-1}{2} R_2$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 3 & 1 & -2 \\ 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & -4 & -1 & 2 \end{array}\right] r_3 = R_3 + 4R_2 \left[\begin{array}{cccc|c} 1 & 1 & 3 & 1 & -2 \\ 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 3 & -6 \end{array}\right] r_3 = \tfrac{1}{3} R_3$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 3 & 1 & -2 \\ 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right] \begin{array}{l} r_2 = R_2 - R_3 \\ r_1 = R_1 - R_3 \end{array} \left[\begin{array}{cccc|c} 1 & 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right] r_1 = R_1 - 3R_2$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right] \longrightarrow$$

parametric solution

$x_1 = -x_2$

x_2 anything

$x_3 = 0$

$x_4 = -2$

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6. Augmented Matrix $\longrightarrow$ the coefficient matrix is the same as in (5) and has RREF of $\left[\begin{array}{cccc} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$ any system is consistent with x_2 a free variable

$$\left[\begin{array}{cccc|c} 1 & 1 & -3 & 1 & 1 \\ 1 & 1 & 1 & -1 & 2 \\ 1 & 1 & -1 & 0 & 0 \end{array}\right]$$

$$\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \to \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \to \begin{bmatrix} 1 \\ -\frac{1}{2} \\ -1 \end{bmatrix} \to \begin{bmatrix} 1 \\ -\frac{1}{2} \\ -3 \end{bmatrix} \to \begin{bmatrix} 1 \\ -\frac{1}{2} \\ -1 \end{bmatrix} \to \begin{bmatrix} 2 \\ \frac{1}{2} \\ -1 \end{bmatrix} \to \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -1 \end{bmatrix}$$

Parametric Solution

$x_1 = \frac{1}{2} - x_2$

x_2 anything

$x_3 = \frac{1}{2}$

$x_4 = -1$