Practice problems for Test 2

1. Label the following statements as TRUE or FALSE.

   (a) Every vector space that is spanned by a finite set has a finite basis.

   (b) A vector space cannot have more than one basis.

   (c) If a vector space has a finite basis, then the number of vectors in every basis is the same.

   (d) The dimension of P_n is n.

   (e) There exists an isomorphism from P_3 to $\mathbb{R}^4$.

   (f) There exists an isomorphism from $\mathbb{R}^n$ to $\mathbb{R}^m$ if and only if $n = m$.

   (g) If F is a linear transformation from $\mathbb{R}^5$ to $\mathbb{R}^3$ and the dimension of the kernel of F is 2, then F is onto.

   (h) Suppose that V is a finite-dimensional vector space, that S_1 is a linearly independent subset of V, and that S_2 is a subset of V that spans V. Then S_1 can contain more vectors than S_2.

   (i) If S is a basis for the vector space V, then every vector in V can be written as a linear combination of vectors in S in only one way.

   (j) If V is a vector space having dimension n, and if S is a subset of V with n vectors, then S is linearly independent if and only if S spans V.

   [redacted]

   (l) If F is an isomorphism from a vector space V to a vector space W, then X is a basis of V if and only if $F(X)$ is a basis of W.

   (m) If A is a 7×4 matrix, then the largest possible dimension of the column space of A is 7.

   (n) If A is a 4×5 matrix, then the largest possible dimension of the column space of A is 4.

   (o) If A is a 8×10 matrix, then the smallest possible dimension of the null space of A is 2.

   (p) If A is a 6×4 matrix, then the smallest possible dimension of the null space of A is 2.

   (q) If A is a 3×2 matrix, then the smallest possible dimension of the null space of A is 0.

   (r) If the null space of a 7×9 matrix A is 5-dimensional, then the dimension of the column space of A is 4.

   (s) If the null space of a 11×9 matrix A is 8-dimensional, then the dimension of the column space of A is 3.

   (t) If A is a 4×8 matrix and the dimension of the row space of A is 3, then the dimension of the null space is 5.

2. Determine which of the following lists of vectors are bases for $\mathbb{R}^3$. If the list is not a basis say which condition(s) in the definition of a basis fails.

$$W = \left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \right) \quad X = \left(\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -5 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} \right)$$

$$Y = \left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) \quad Z = \left(\begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} \right)$$

3. Find bases for the column space and the null space of the matrix

$$M = \begin{bmatrix} 1 & 1 & 5 & 1 & 4 \\ 2 & -1 & 1 & 2 & 2 \\ 3 & 0 & 6 & 0 & -3 \end{bmatrix}$$

4. Find bases for the row space, column space, and null space of the matrix

$$N = \begin{bmatrix} 2 & 1 & 4 & -1 & 4 \\ 1 & -1 & 5 & 1 & -1 \\ -1 & 2 & -7 & 0 & 1 \\ 2 & -1 & 8 & -1 & 2 \end{bmatrix}$$

5. (Going down) Let $X = (\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4)$, where

$$\vec{v}_1 = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \quad \text{and} \quad \vec{v}_4 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

(a) Show that X spans $\mathbb{R}^3$.

(b) Show that X is linearly dependent, and find a specific linear dependence relation between the vectors in X.

(c) Solve the linear dependence relation to express the vector $\vec{v}_1$ as a linear combination of the other vectors in X.

(d) Explain, without any further calculations, why $X' = (\vec{v}_2, \vec{v}_3, \vec{v}_4)$ is a basis of $\mathbb{R}^3$.

6. (Going up) Let $Y = (\vec{w}_1, \vec{w}_2, \vec{w}_3)$, where

$$\vec{w}_1 = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{w}_2 = \begin{bmatrix} 4 \\ 4 \\ -2 \\ 6 \end{bmatrix}, \quad \text{and} \quad \vec{w}_3 = \begin{bmatrix} 2 \\ 6 \\ -3 \\ 5 \end{bmatrix}$$

(a) Show that Y is linearly independent.

(b) Find a vector $\vec{w}_4$ not in the span of Y.

(c) Explain, without any further calculations, why $Y' = (\vec{w}_1, \vec{w}_2, \vec{w}_3, \vec{w}_4)$ is a basis for $\mathbb{R}^4$.

7. Express $q(x) = 5 + 9x + 5x^2$ as linear combination of $p_1(x) = 2 + x + 4x^2$, $p_2(x) = 1 - x + 3x^2$, and $p_3(x) = 3 + 2x + 5x^2$.

8. Determine whether the following polynomials span P_2.

$$p_1(x) = 1 + 2x - x^2 \qquad p_2(x) = 3 + x^2$$
$$p_3(x) = 5 + 4x - x^2 \qquad p_4(x) = -2 + 2x - 2x^2$$

9. Which of the following list of vectors are bases for P_2?

(a) $X_1 = (3 + x + x^2, 2 - x + 3x^2, 1 - 3x + 5x^2)$

(b) $X_2 = (6 - x^2, 1 + x + 4x^2)$

(c) $X_3 = (1 + x, 1 + x + x^2, 1 - x^2)$

(d) $X_4 = (1 + 3x + 3x^2, x + 4x^2, 5 + 6x + 3x^2, 7 + 2x - x^2)$

10. Let $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation, and suppose

$$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}1\\1\end{bmatrix}, \quad T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}3\\0\end{bmatrix}, \quad T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}4\\-7\end{bmatrix}$$

(a) Write down the standard matrix of T.

(b) Find $T\left(\begin{bmatrix}1\\3\\8\end{bmatrix}\right)$ (c) Find $T\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right)$

11. Let $D\colon P_1 \to P_1$ be the linear transformation defined by

$$D(p(x)) = (1 + 2x)p'(x) - 3p(x),$$

and let $X = (p_1(x), p_2(x))$ and $Y = (q_1(x), q_2(x))$, where

$$p_1(x) = 1 + x, \quad p_2(x) = 1 + 2x, \quad q_1(x) = 4 - x, \quad q_2(x) = -3 + x.$$

(a) Find the matrix of D with respect to the basis X.

[redacted]

12. Label the following statements as TRUE or FALSE.

(a) If A is an $n \times n$ matrix whose columns span $\mathbb{R}^n$, then A is invertible.

[redacted]

(c) If A is a noninvertible $n \times n$ matrix, then $\operatorname{Col} A = \mathbb{R}^n$.

(d) If A is an invertible $n \times n$ matrix, then $\operatorname{Nul} A = \{\vec{0}\}$.

[redacted]

(g) If A is an $n \times n$ matrix whose columns are linearly dependent, then A is invertible.

(h) The set of al vectors of the form $(a, 2a + b, 2b + 3)$ is a subsapce of $\mathbb{R}^3$.

[redacted]

(j) The zero vector is a linear combination of any nonempty set of vectors.

[redacted]

(l) If S is a linearly dependent set, then each vector in S is a linear combination of other vectors in S.

(m) Any set containing the zero vector is linearly dependent.

(n) Subsets of linearly dependent sets are necessarily linearly dependent.

(o) Subsets of linearly independent sets are necessarily linearly independent.

13. Find the standard matrix for the linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ that maps a point (x_1, x_2) into:

(a) its reflection about the line $x_2 = -x_1$

(b) its projection on the x_1-axis

(c) its projection on the x_2-axis

(d) its rotation counterclockwise through and angle of $\pi/6$

(e) its rotation clockwise through and angle of $\pi/3$

For each part, use the matrix you have obtained to compute $T(2, 1)$.

14. Suppose that T is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$, which satisfies $T(1, 2) = (1, 3, 5)$ and $T(2, 7) = (-1, 1, 1)$.

(a) Write $(1, 0)$ and $(0, 1)$ as linear combinations of the vectors $(1, 2)$ and $(2, 7)$.

(b) Use the results of part (a) to evaluate $T(1, 0)$ and $T(0, 1)$.

(c) Write down the standard matrix of T.

15 Let $G\colon \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation given by the formula

$$G(x_1, x_2, x_3, x_4) = (x_1 - 2x_3 - x_4, x_2 + 7x_3 - x_4, x_1 + x_2 + 5x_3)$$

(a) Write down the standard matrix of G.

(b) Find a nontrivial vector in the kernel of G.

(c) Explain why G is onto.

16 Let $X = (\vec{v}_1, \vec{v}_2, \vec{v}_3)$, where $\vec{v}_1 = (2,1,4)$, $\vec{v}_2 = (5,4,7)$, and $\vec{v}_3 = (3,2,5)$.

(a) Does X span all of $\mathbb{R}^3$?

(b) If not, find a homogeneous linear system (possibly with only one equation) whose solution set is the span of X.

(c) Is X linearly independent?

(d) If not, find a linear dependence relation on X.

17 Let $Y = (\vec{w}_1, \vec{w}_2, \vec{w}_3, \vec{w}_4)$, where $\vec{w}_1 = (1,2,5,-7)$, $\vec{w}_2 = (3,1,-5,9)$, $\vec{w}_3 = (1,-1,-7,11)$, and $\vec{w}_4 = (1,0,-3,5)$.

(a) Does Y span all of $\mathbb{R}^4$?

(b) If not, find a vector in $\mathbb{R}^4$ that is not in Span(Y).

(c) Is Y linearly independent?

(d) If not, find a linear dependence relation on Y.

18 Which of the following lists of vectors in $\mathbb{R}^3$ are linearly independent? These can all be done "by inspection", i.e. without any work.

(a) $X_a = \big((2,1,4),(0,0,0),(2,10,-4)\big)$

(b) $X_b = \big((3,1,-2),(2,-1,5),(12,4,-8)\big)$

(c) $X_c = \big((6,0,-1),(1,1,4)\big)$

(d) $X_d = \big((-2,6,4),(3,-9,-6)\big)$

(e) $X_e = \big((1,3,3),(0,1,4),(5,6,3),(7,2,-1)\big)$