Homework 25 MATH 304 Section 3 Solution

Assigned: Wednesday, November 26.
Potentially Collected: Wednesday, December 3.

1. Which of the following matrices are diagonalizable?

   (a) $\begin{bmatrix} 1 & 4 \\ 1 & -2 \end{bmatrix}$

   (b) $\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$

   (c) $\begin{bmatrix} 1 & 1 & -2 \\ 4 & 0 & 4 \\ 1 & -1 & 4 \end{bmatrix}$

$P_B(\lambda) = \det\left(\begin{bmatrix} 1-\lambda & 0 \\ -2 & 1-\lambda \end{bmatrix}\right) = (1-\lambda)^2$

$E_1 = Nul\left(\begin{bmatrix} 0 & 0 \\ -2 & 0 \end{bmatrix}\right) = Span\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$

Since $\dim(E_1) = 1$, the sum of the dimensions of the eigenspaces for B is $1 \neq 2$.

B is ~~NOT~~ diagonalizable.

2. Let $A = \begin{bmatrix} 3 & -5 \\ 1 & -3 \end{bmatrix}$. Compute A^9 by finding a matrix P such that $P^{-1}AP$ is a diagonal matrix D and show that $A^9 = PD^9P^{-1}$.

$$P_A(\lambda) = \det\left(\begin{bmatrix} 1-\lambda & 4 \\ 1 & -2-\lambda \end{bmatrix}\right) = (1-\lambda)(-2-\lambda) - 4$$

$$= -2 + \lambda + \lambda^2 - 4 = \lambda^2 + \lambda - 6 = (\lambda-2)(\lambda+3)$$

The 2x2 matrix has 2 distinct eigenvalues, therefore A is diagonalizable.

$$P_C(\lambda) = \det\left(\begin{bmatrix} 1-\lambda & 1 & -2 \\ 4 & -\lambda & 4 \\ 1 & -1 & 4-\lambda \end{bmatrix}\right) = (1-\lambda)(-1)^{1+1}\det\left(\begin{bmatrix} -\lambda & 4 \\ -1 & 4-\lambda \end{bmatrix}\right) + (1)(-1)^{1+2}\det\left(\begin{bmatrix} 4 & 4 \\ 1 & 4-\lambda \end{bmatrix}\right) + (-2)(-1)^{1+3}\det\left(\begin{bmatrix} 4 & -\lambda \\ 1 & -1 \end{bmatrix}\right)$$

$$= (1-\lambda)\left(-\lambda(4-\lambda)+4\right) - \left(4(4-\lambda)-4\right) - 2(-4+\lambda) = -\lambda^3 + 5\lambda^2 - 6\lambda$$

$$= -\lambda(\lambda^2 - 5\lambda + 6) = -\lambda(\lambda-3)(\lambda-2)$$

The 3x3 matrix has 3 distinct eigenvalues, therefore C is diagonalizable.