10/2/2014 Version A

Math 304: Linear Algebra Midterm Exam 1

Solution

Problem	1	2	3	4	5	6	Total
Full Score	20	20	20	20	20	20	120
Your Score							

- Read all problems before beginning and try to work from easiest to hardest.
- In order to get credit, you must show all of your work.
- **NO** calculators of any kind! **NO** cell phones!
- Check to make sure that your exam has six (6) pages and six (6) questions.

1. **Clearly** circle "True" or "False" for each of the following problems. Circle "True" only if the statement is always true. No explanation necessary.

TRUE FALSE (a) Let A be an $m \times n$ matrix with $m > n$. Then any row echelon form contains at least $m - n$ zero rows.

TRUE FALSE (b) An $m \times n$ matrix has m rows and n columns.

TRUE FALSE (c) The matrix $\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$ is in reduced row echelon form.

TRUE FALSE (d) Suppose that A is a 5×3 matrix. Then $A\vec{x} = \vec{0}$ has infinitely many solutions.

TRUE FALSE (e) The rank of an 11×7 matrix is greater or equal to 7.

TRUE FALSE (f) Every elementary matrix is nonsingular.

TRUE FALSE (g) $(AB^T)^T$ is always equal to $A^T B$ for all matrices A and B such that AB^T is defined.

TRUE FALSE (h) The matrix $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ is in diagonal form.

TRUE FALSE (i) Let A, B be $n \times n$ matrices and $rank(A) = n$. Then the matrix equation $AX = B$ is always solvable.

TRUE FALSE (j) Let $\vec{u}$ and $\vec{v}$ be different solutions of the nonhomogeneous system $A\vec{x} = \vec{b}$. Then $\vec{u} - \vec{v}$ is a nontrivial solution of the associated homogeneous system.

2. Suppose that

$$M = \left[\begin{array}{ccccc|c} 0 & 1 & -1 & 1 & 2 & 5 \\ 0 & 2 & -2 & 2 & 6 & 18 \\ 0 & -1 & 4 & 5 & -2 & -11 \end{array}\right]$$

is the augmented matrix of a system of linear equations in the variables x_1, x_2, x_3, x_4, x_5.

a) Bring the matrix M into reduced row echelon form, indicating all elementary row operations.

$$\begin{array}{l} r_2 = R_2 - 2R_1 \\ r_3 = R_3 + R_1 \end{array} \left[\begin{array}{ccccc|c} 0 & 1 & -1 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 & 2 & 8 \\ 0 & 0 & 3 & 6 & 0 & -6 \end{array}\right] \quad \begin{array}{l} r_2 = R_3 \\ r_3 = R_2 \end{array} \left[\begin{array}{ccccc|c} 0 & 1 & -1 & 1 & 2 & 5 \\ 0 & 0 & 3 & 6 & 0 & -6 \\ 0 & 0 & 0 & 0 & 2 & 8 \end{array}\right]$$

$$\begin{array}{l} r_2 = \tfrac{1}{3}R_2 \\ r_3 = \tfrac{1}{2}R_3 \end{array} \left[\begin{array}{ccccc|c} 0 & 1 & -1 & 1 & 2 & 5 \\ 0 & 0 & 1 & 2 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right] \quad r_1 = R_1 - 2R_3 \left[\begin{array}{ccccc|c} 0 & 1 & -1 & 1 & 0 & -3 \\ 0 & 0 & 1 & 2 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right]$$

$$r_1 = R_1 + 2R_2 \left[\begin{array}{ccccc|c} 0 & 1 & 0 & 3 & 0 & -5 \\ 0 & 0 & 1 & 2 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right]$$

b) Which variables are the basic variables?

x_2, x_3, x_5

c) Which variables are the free variables?

x_1, x_4

d) What is the rank of M?

$\text{rank}(M) = 3$

e) List the columns of M which are pivot columns.

$$\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 6 \\ -2 \end{bmatrix}$$

f) If the system is consistent, write its solution in parametric form.

$x_2 = -5 - 3x_4$ $\qquad$ x_1 anything

$x_3 = -2 - 2x_4$ $\qquad$ x_4 anything

$x_5 = 4$

3. Given a matrix

$$M = \left[\begin{array}{ccccc|c} 1 & 2 & 0 & 3 & a & b \\ 0 & 0 & 1 & 1 & c & d \\ 0 & 0 & 0 & 0 & e & f \end{array}\right]$$

representing the augmented matrix of a system of equations in reduced row echelon form. Compute the following by filling in the blanks.

(a) For $a =$ anything, $b =$ 0, $c =$ anything, $d =$ 0, $e =$ 0, $f =$ 1, the matrix M represents the reduced row echelon form of an inconsistent system of equations.

$\hookrightarrow f=1, b=d=0$ $\qquad$ $\hookrightarrow e=0, f\neq 0$

The pivots are located at m_{11}, m_{23}, m_{36}. (Give your answer in the form m_{ij}.)

The rank of the coefficient matrix is 2.

The rank of the augmented matrix is 3.

(b) For $a =$ 0, $b =$ anything, $c =$ 0, $d =$ anything, $e =$ 1, $f = 2$, the matrix M is the augmented matrix of a consistent nonhomogeneous system in reduced row echelon form.

$\hookrightarrow e\neq 0$ $\qquad$ $\hookrightarrow$ not all zero; b, d, f $\qquad$ $\hookrightarrow e=1, a=c=0$

The pivots are located at m_{11}, m_{23}, m_{35}.

The rank of the augmented matrix is 3.

The rank of the coefficient matrix is 3.

(c) For $a = 1$, $b =$ 0, $c = 1$, $d =$ 0, $e =$ 0, $f =$ 0, the matrix M is the augmented matrix of a homogeneous system of rank 2 in reduced row echelon form.

$\hookrightarrow b=d=f=0$ $\qquad$ $\hookrightarrow e=0$

The complete solution in parameterized form is

$x_1 =$ ______, $x_2 =$ ______, $x_3 =$ ______, $x_4 =$ ______, $x_5 =$ ______.

$x_1 = -2x_2 - 3x_4 - x_5$ $\qquad$ $x_3 = -x_4 - x_5$

x_2 anything $\qquad$ x_4, x_5 anything

4. Consider the matrices $A = \begin{bmatrix} 1 & 2 \\ \frac{1}{2} & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 & -4 \\ -1 & \frac{1}{2} & 2 \end{bmatrix}$. For each of the following operations, either do the indicated calculations or explain why it is not defined.

(i) $A + B$

Undefined

(ii) $A \cdot B$

$(2\times2)(2\times3)$ $\downarrow$ results in 2×3 matrix

$$= \begin{bmatrix} [1 \ 2]B \\ [\frac{1}{2} \ 1]B \end{bmatrix} = \begin{bmatrix} 1[2 \ -1 \ -4] + 2[-1 \ \frac{1}{2} \ 2] \\ \frac{1}{2}[2 \ -1 \ -4] + 1[-1 \ \frac{1}{2} \ 2] \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(iii) $B \cdot A$

$(2\times3)(2\times2)$

Undefined

(iv) $(A \cdot B)^2$

Undefined

(v) $2A^T + 8A^T \cdot B$

$$8A^TB = \begin{bmatrix} 8 & 4 \\ 16 & 8 \end{bmatrix} B = \begin{bmatrix} [8 \ 4]B \\ [16 \ 8]B \end{bmatrix} = \begin{bmatrix} 8[2 \ -1 \ -4] + 4[-1 \ \frac{1}{2} \ 2] \\ 16[2 \ -1 \ -4] + 8[-1 \ \frac{1}{2} \ 2] \end{bmatrix}$$

$$= \begin{bmatrix} 12 & -6 & -24 \\ 24 & -12 & -48 \end{bmatrix}$$

$2A^T + 8A^TB$

$= \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} + \begin{bmatrix} 12 & -6 & -24 \\ 24 & -12 & -48 \end{bmatrix}$ undefined

(vi) $B^T \cdot A^T$.

$$= \begin{bmatrix} 2 & -1 \\ -1 & \frac{1}{2} \\ -4 & 2 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

(vii) How are the matrices $A \cdot B$ and $B^T \cdot A^T$ related? Justify your answer.

$(AB)^T = B^T A^T$

5. Let A be the vector space consisting of column vectors of length 4 and let B be the vector space of column vectors of length 3. Consider the function $f : A \longrightarrow B$ given by

$$f\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 + 2x_2 + 2x_3 + 6x_4 \\ 2x_1 + 2x_2 + x_3 + 3x_4 \\ x_1 + 2x_2 + x_3 + 3x_4 \end{bmatrix}$$

$f: \mathbb{R}^4 \to \mathbb{R}^3$

(a) What is the domain of f? What is the codomain?

$\mathbb{R}^4$ $\quad$ $\mathbb{R}^3$

(b) Determine $f(\vec{e_i})$ for $i = 1, 2, 3, 4$ for the standard basis $\{\vec{e_1}, \vec{e_2}, \vec{e_3}, \vec{e_4}\}$ of $\mathbb{R}^4$ written as column vectors.

$f(\vec{e_1}) = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \quad f(\vec{e_2}) = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}, \quad f(\vec{e_3}) = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \quad f(\vec{e_4}) = \begin{bmatrix} 6 \\ 3 \\ 3 \end{bmatrix}$

(c) Using (b), write down the standard matrix M such that $f(\vec{x}) = M \cdot \vec{x}$.

$$M = \begin{bmatrix} 2 & 2 & 2 & 6 \\ 2 & 2 & 1 & 3 \\ 1 & 2 & 1 & 3 \end{bmatrix}$$

(d) Determine the rank of M.

$r_1 = \frac{1}{2}R_1 \begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & 2 & 1 & 3 \\ 1 & 2 & 1 & 3 \end{bmatrix} \begin{matrix} r_3 = R_3 - R_1 \\ r_2 = R_2 - 2R_1 \end{matrix} \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 0 & -1 & -3 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{matrix} r_2 = R_3 \\ r_3 = R_2 \end{matrix} \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & -3 \end{bmatrix}$

$\operatorname{rank}(M) = 3$

(e) Is the function f one-to-one? Explain.

NO

$\operatorname{rank}(M) = 3 \neq 4 = \#$ columns

(f) Is the function f onto? Explain. YES

$\operatorname{rank}(M) = 3 = \#$ rows

6. Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 2 & 2 & 0 & 1 \end{bmatrix}$.

(a) Find A^{-1} and check your result. Solve $AX = I_2$

$$[A \,\vdots\, I_2] = \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 2 & 5 & 0 & 1 \end{array}\right] \quad r_2 = R_2 - 2R_1 \quad \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{array}\right] \quad r_1 = R_1 - 2R_2$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 5 & -2 \\ 0 & 1 & -2 & 1 \end{array}\right] = [I_2 \,\vdots\, A^{-1}]$$

$$A^{-1} = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}$$

Check: $AA^{-1} = A^{-1}A = I_2$

(b) Use your work from part (a) to express A^{-1} and then A as a product of elementary matrices.

$r_2 = R_2 - 2R_1$ corresponds to multiplication by $E_1 = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$

$r_1 = R_1 - 2R_2$ " " " $E_2 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$

$$A^{-1} = E_2 E_1$$

$$A = E_1^{-1} E_2^{-1} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

(c) Solve the matrix equation $AX = B$ using A^{-1} from part (a).
(You must use A^{-1}, not any other method.)

$$AX = B \Rightarrow A^{-1}AX = A^{-1}B$$

$$\Rightarrow X = A^{-1}B$$ ← be very careful with commutativity!

$$X = A^{-1}B = \begin{bmatrix} 1 & 1 & 0 & -7 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$