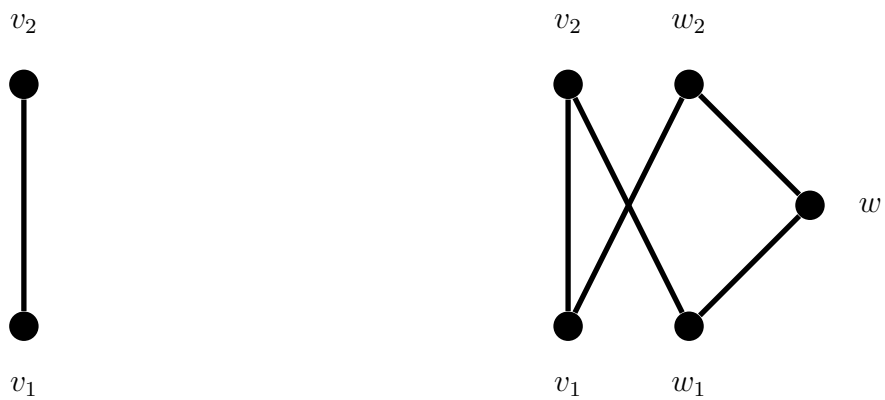
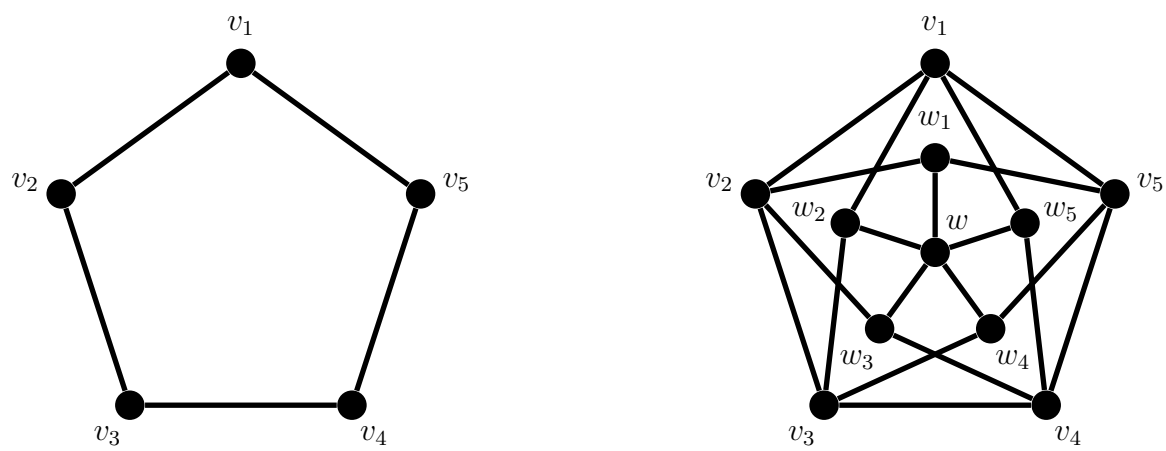
Quiz 6

April 24th, 2017

1) We describe a certain construction on a labeled graph. For a (simple) graph G with vertices $v_1, \ldots, v_n$, create a new graph G' as follows: add vertices $w_1, \ldots, w_n$ drawing an edge between w_i and every vertex in $N(v_i)$ (recall this is the *neighborhood* of v_i, the set of vertices in G adjacent to it). Then add one final vertex w which is adjacent to all of the w_i's. Here is this construction on K_2:



And on C_5:



a) Do this construction for:

i) K_3

ii) C_4

iii) P_5

iv) $K_{3,3}$

b) Prove that if G has no complete subgraph of size $k > 2$, then neither does G'. (HINT: How many w_i's can be a part of this subgraph? Can w?)

c) Prove that if $\chi(G) = d$, then $\chi(G') = d + 1$. (HINT: Show $\chi(G') > d$, and then explain how to color it with $d + 1$ colors).

d) Explain what statement made in the book (the authors claim something interesting but do not prove it) the previous two parts prove.

2) For a labeled graph G, define a function $P_G(d)$ (for $d \in \mathbb{N}$ to be the number of proper colorings of G with d colors (if $d < \chi(G)$, then $P_G(d) = 0$ for example). This function is called the *chromatic polynomial* of G (proving it is a polynomial is way beyond us, but we will be able to do some stuff, and a lot of it will look familiar).

a) Figure out the following values:

i) $P_{K_4}(3)$
ii) $P_{K_4}(4)$
iii) $P_{K_4}(5)$
iv) $P_{P_6}(2)$
v) $P_{P_6}(3)$
vi) $P_{C_5}(3)$
vii) $P_{C_6}(3)$
viii) $P_{W_6}(3)$

b) Explain why, if G has two components G_1 and G_2, $P_G(d) = P_{G_1}(d) \cdot P_{G_2}(d)$.

c) If T is a tree with n vertices, find $P_T(d)$ (it will be a polynomial involving n and d) (HINT: Exercise 1.8.26, which was part of your very first homework, may give you some ideas).

d) Find $P_{W_n}(3)$ (HINT: you will have two cases).

e) Explain why, for any edge $e \in E(G)$, $P_G(d) = P_{G-e}(d) - P_{G/e}(d)$ (HINT: What is the difference between P_G and P_{G-e}?).

3) SHORT ANSWER: Briefly explain AND draw an example or a counterexample.

a) Two graphs G_1 and G_2 which are not "essentially the same" can have the same chromatic polynomial.

b) Two connected planar graphs G_1 and G_2 which are not "essentially the same" can have the same planar dual.

c) Two connected graphs G_1 and G_2 which are not "essentially the same" can have the same number of spanning trees.

d) A graph can have completely disjoint perfect matchings (in the sense that no edge of one matching is an edge of the other).

e) If a graph has a cycle of length k, then it has at least k spanning trees.

rules:

1. Work out the problems on scratch paper, and transfer your neat and clean solutions onto separate paper.
2. This is due at 9:30AM on Wednesday, April 26th. Make sure your name is on what you turn in.
3. While working in class, the quiz is open book and open notes, but cell phones may not be out.
4. Failure of any group member to follow the rules will result in a score of zero for every group member.