Math 314: Discrete Mathematics
Homework 7 Solutions

1. Problem 10.1.2. (Your graphs must not be the ones in the back): Show by examples that the conditions formulated in the theorem cannot be dropped:

   a) A nonbipartite graph in which every node has the same degree need not contain a perfect matching.

   Solution: Any odd cycle will do (though C_3 is the example in the back), and there are certainly others.

   b) A bipartite graph in which every node has positive degree (but not all the same) need not contain a perfect matching.

   Solution: Again, there are many; $K_{1,2}$ is a star (the example in the back), so... $K_{2,3}$.

2. a) Problem 10.4.10.: Does the graph in figure 10.10 have a perfect matching?

   Solution: No, it does not: label each vertex with an ordered pair representing its row and its column in the grid. Then, assign a number to each vertex that is the sum of its two coordinates. Each edge must go from a vertex with an even number to a vertex with an odd number; hence the graph is bipartite. However, the two parts have different numbers of vertices (32 vs. 30), hence no perfect matching can exist.

   b) Surprise surprise, you've done this problem before too. Identify the problem, and *carefully* explain how the two are the same.

   Solution: Problem 2 on Homework 2 asks about tiling a truncated chessboard with dominos, and is the same. A vertex in the graph is a square on the chessboard, and the edges represent adjacencies of the squares. A matched edge is exactly what a domino would cover, and so an additional solution to *a*) is "the chessboard cannot be completely tiled, so the graph has no perfect matching."

3. Problem 10.4.5.: Does there exist a bipartite graph with degrees $3, 3, 3, 3, 3, 3, 3, 3, 3, 5, 6, 6$?

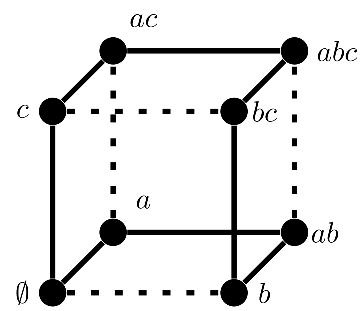
   Proof: No. The edges all go from one side to the other, so the sum of the degrees must be the same on both sides (as a necessity). Then we must be able to partition those degrees into two sets whose sums are 22. The 5 must go somewhere, and so the other vertices on its side must have degrees summing to 17. However, all the remaining vertices in question have degree 3 or 6, so the sum of any collection of these will be a multiple of 3, which 17 is not.

4. If a bipartite graph has $k > 0$ connected components, each has a nonempty vertex set, all the vertices are labeled, and one component has at least one edge, how many "bipartitions" of the vertices are there? That is, how many ways are there of writing the vertex set as $A \cup B$ such that $A \cap B = \emptyset$ and the induced subgraphs on A and on B have no edges? Does your formula still work if I don't require that one component has at least one edge? Why or why not?

   Solution: We prove a lemma: that for a connected bipartite graph, the bipartition is unique up to switching the entire sides. So let G be connected and bipartite. Since G is connected, it has a spanning tree. Choose any vertex in G, and record the (unique) distances from this vertex to every other. One side of the bipartition of G will be all vertices for which this distance is even, the other all vertices for which this distance is odd. For a different spanning tree, the distances may be different, but the parity will not be (two paths of opposite parity would give rise to an odd cycle, which G cannot have). As such, the bipartition is unique up to which vertex we choose to begin with, which may force all of one side to swap with all of the other. This proves the lemma.

   Then for each component, there are precisely 2 bipartitions, and so for k components there will be 2^k ways to select an A and a B. An isolated vertex can be placed into either A or B, but if all vertices are isolated (i.e. if we don't insist that there is at least one edge) we run the risk of placing all vertices into A and none into B (or vice versa), which is not a legitimate bipartition (in this case, the formula is $2^k - 2$).

5. Problem 10.4.11.: Draw a graph whose vertices are the subsets of a, b, c and for which two vertices are adjacent if and only if they are subsets that differ in exactly one element.



a) What is the number of edges and vertices in this graph? Can you name this graph?

   Solution: It has 8 vertices, and 12 edges, and yes, I can name it (it's a cube).

b) Is this graph connected? Does it have a perfect matching? Does it have a Hamilton cycle?

   Solution: By inspection, it's connected. The dashed edges are a perfect matching, and their complement, the solid edges, form a Hamilton cycle (I used Petersen's Theorem on the Hamilton cycle to get the perfect matching).

REMINDER: These represent possible solutions to each problem. The solution methods are not necessarily unique, and there are likely other correct solutions.