Math 314: Discrete Mathematics
Homework 6 Solutions

1. By straightforward algebra, for any k and n with $0 \leq k \leq n$,

$$\binom{n}{2} = \binom{k}{2} + k(n-k) + \binom{n-k}{2}$$

(you just manipulate the right hand side). Using things you know about complete graphs, prove this fact without using any algebra whatsoever (explain why the left and right hand sides are equal).

Proof: $\binom{n}{2}$ is the number of edges in the complete graph on n vertices. Drawing k vertices on one side of a picture and $n-k$ vertices on the other, and then drawing every edge yields the right-hand side.

2. Problem 8.5.4.: Prove that if a tree has a node of degree d, then it has at least d leaves.

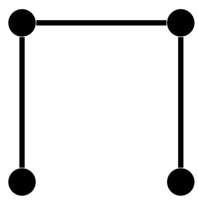
Proof: Consider a vertex v of degree d in a tree. By definition, every edge of a tree is a cut edge, so in particular, every edge incident with v is a cut edge. Deleting v then yields a forest with exactly d components. Each component is a tree, so by Theorem 8.2.1, each component has at least two leaves. They all have a vertex that is adjacent to v in the original tree, which might now be a leaf, but then they all still have at least one other, which is a leaf in the original tree.

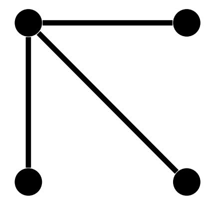
3. Are there any graphs G such that G and $\overline{G}$ are both trees? If so, what are they? Prove your solution (that there are none, or there are no others).

Proof: If G has no vertices, G is the empty graph, which is its own complement, so that is one such graph. If G has at least one vertex and is connected, we may use the fact that it is a tree if and only if it has one fewer edge than vertex. So if G has n vertices, it has $n-1$ edges. The number of edges in K_n is $\binom{n}{2}$, and $\overline{G}$ has exactly the edges of K_n that aren't in G, so $\overline{G}$ has $\binom{n}{2} - (n-1)$ edges. But for $\overline{G}$ to also be a tree, it must have $n-1$ edges. Then

$$\begin{aligned}\binom{n}{2} - (n-1) &= n-1 \\ \binom{n}{2} &= 2(n-1) \\ n(n-1) &= 4(n-1) \\ (n-4)(n-1) &= 0,\end{aligned}$$

and so G can have either 1 vertex or 4 vertices. The tree on 1 vertex is also its own complement, and so it is another. There are two essentially different trees on 4 vertices:





The first has a tree as its complement (bringing our total to three graphs), but the second does not.

4. For a tree T, define $P[u, v]$ to be the subgraph that is the (unique) path between u and v. Prove that, for three distinct vertices u, v, and w in a tree, that the intersection $P[u, v] \cap P[u, w] \cap P[v, w]$ consists of a single vertex (this is called the *median* of u, v, and w).

   Proof: We first claim the intersection is not empty: $P[u, v] \cup P[v, w]$ is a walk from u to w; deleting all edges and vertices which get used more than once, this is a path from u to w. If this does not intersect $P[u, w]$, then the union of the two is a cycle, which contradicts the definition of a tree. But this is true of any two of the vertices u, v, w, and so our claim is proved. Now we claim it cannot have more than one vertex. If it did, they would be on $P[u, v]$ and $P[v, w]$, whence the paths between these two would not be unique as they are in a tree. So we are done.

5. Problem 9.2.3.: Prove that if all edge-costs are different, then there is only one cheapest (*spanning*) tree.

   Proof: Without the word "spanning" this is false, so I assume that's what they meant. The greedy algorithm for this (choosing the cheapest edge that does not create a cycle, and repeat until there are edges numbering one fewer than the number of vertices) is known to produce a cheapest spanning tree. If all edge-costs are different, then the greedy algorithm never makes a choice, and so its output is unique. Then the unique output is the only cheapest spanning tree.

REMINDER: These represent possible solutions to each problem. The solution methods are not necessarily unique, and there are likely other correct solutions.