Exam 1 Review

1) The winning powerball numbers consist of a set of 5 numbers chosen from [69] and a special 6^{th} number (the "powerball") chosen from a separate collection of [26]. To win the grand prize, you must match all 5 numbers (not necessarily in order) and the powerball.

   a) How many possible sets of lottery numbers are there, and what are the odds of winning the grand prize?

   b) What is the probability of winning, given that you know you got the powerball correct?

   c) Suppose you had to get the 5 numbers in order. How much harder would it be to win the grand prize?

2) The movie "Cheaper by the Dozen" is about a family with 12 children.

   a) Prove at least 2 of the children were born on the same day of the week.

   b) Prove at least 2 family members (including the parents) were born in the same month.

   c) If the children share 4 bedrooms in the house, prove that at least one bedroom is shared by at least 3 children.

   d) True or False? Answer (with justification) each of the following.

      * Each of the 4 bedrooms is shared by 3 children.

      * The number of bedrooms occupied by more than 3 children is equal to the number of bedrooms occupied by fewer than 3 children.

3) Prove that if you write down *any* 10 integers, then some consecutive (nonempty) subset of them adds to a multiple of 10.

4) (Hard) Prove that there is a power of 3 that ends in 001.

5) (Hard) Prove that, in any group of 6 people, there is a group of 3 mutual friends or a group of 3 mutual strangers.

6) Use induction to prove each of the following

   a) $n^3 + 2n$ is divisible by 3 for $n > 0$.

   b) Prove that

   $$\left(\sum_{i=1}^{n} i\right)^2 = \sum_{i=1}^{n} i^3.$$

   You may use the fact that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ after the inductive hypothesis is invoked, but not before.

   c) Prove that $F_n \leq 2^{n-1}$ for $n > 0$ (recall F_n is the n^{th} Fibonacci number, and $F_0 = 0$ is not a case).

   d) (Hard) $3^n > n^2$ for $n > 0$.

   e) (Hard) Prove (by induction on n) that, so long as $x \geq -1$, $(1+x)^n \geq 1 + nx$ for $n > 0$. Figure out where you use the assumption that $x \geq -1$.

7) Prove that the number of ways to tile a $2 \times n$ chessboard with 2×1 dominoes is F_{n+1}

8) How many ways can a poker hand (5 cards) be selected from a regular deck (52 cards) so that the hand contains at least one card from each of the 4 suits?

9) There are 600 students taking at least one of calc 3, linear algebra, and discrete mathematics. There are 315 students in calc 3, 288 in linear algebra, and 80 in discrete. There are 35 students taking both calc 3 and linear, 22 students taking both calc 3 and discrete, and 34 students taking both linear and discrete. How many students are taking all 3 courses?

10) How many 7-digit natural numbers without a 0 anywhere do not contain 121 as a string of three consecutive digits anywhere?

11) Which natural numbers $0 \leq n \leq 19$ have multiplicative inverses mod 20?

12) What is the multiplicative inverse of 9 mod 37?

13) Solve $17x \equiv 7 \bmod 35$.

notes:

1. This review does not claim to be (nor is it) complete. Anything we have done in class, anything similar to the homework or quizzes, and anything similar to questions in the book is fair game.

2. The test will cover all of chapters 1-6. To be adequately prepared you should be ready to answer questions from any of that.

3. For additional practice, please refer to the homework assignments, the problems in the book, your notes, and the quizzes.