Notes on Characterizing Trees

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20 April 2020

Definition. A graph is *acyclic* when it has no cycle subgraphs. A *tree* is a connected acyclic graph.

Proposition. *Let G be a graph. The following are equivalent.*

1. *Graph G is a tree.*
2. *Graph G is a connected acyclic graph.*
3. *Graph G is minimally connected (i.e. G is connected but for all $e \in E(G)$ we have $G \backslash e$ is not connected).*
4. *Graph G has a unique pair connecting each pair of vertices.*
5. *Graph G is maximally acyclic (i.e. G is acyclic, but any new edge e creates a cycle in $G \cup e$).*

Proof. Let G be a graph.

$2 \implies 3$: Assume G is connected and acyclic. Let $e \in E(G)$ and consider $G \setminus e$. Let $u, v \in V(G)$ be the ends of e in G. We know this is a path from u to v in G. Assume to the contrary, there is a path P from u to v in $G \setminus e$ such that P does not contain e; thus (u, P, v, e, u) is a cycle in G, contradicting the assumption that G is acyclic. Thus $G \setminus e$ is disconnected. Hence G is minimally connected.

$3 \implies 4$: Assume G is minimally connected. Let $u, v \in V(G$. As G is connected, there is a path P connecting u to v in G. If $u = v$ this is trivial. Otherwise there is an edge e in P. $G \setminus e$ is disconnected so every path from u to v in G crosses e. As e was arbitrary, P is the unique path connecting u to v.

$4 \implies 5$: Assume every pair of vertices in G has a unique path connecting them. Suppose we are given an additional edge e. Let $u, v \in V(G)$ denote the ends of e. We know there is a unique path P in G connecting u to v; thus P does not contain e, and we see (u, P, v, e, u) is a cycle in $G \cup e$. Hence G is maximally acyclic.

$5 \implies 2$: Assume G is maximally acyclic. Let $u, v \in V(G)$. Add an edge e connecting u to v; this creates a cycle C in G by our assumption that G is maximally acyclic. Now C must use e lest G contains a cycle. Thus removing e from C yields a path $P = C \setminus e$ which has u and v as its ends. Hence G is connected. □

Remark. This characterization applies to all trees; for finite trees we can extend the result.

Definition. A *leaf* of a tree is a vertex of degree 1.

Proposition. *Every finite tree with at least two vertices has a leaf.*

Proof. Let T be a finite tree with $n \in \mathbb{Z}_{\geq 2}$ vertices, and assume to the contrary that no vertex of G has degree 1. Now $\deg(v) \geq 2$ for all $v \in V(T)$ as T is connected and has at least two vertices. Let v_0v_1 be any edge of T and define $W_1 = (v_0, v_1)$. Having defined $W_k = (v_0, v_1, \ldots, v_k)$ with $v_{i-1}v_i \in E(G)$ for each $i \in [k]$, note that $\deg(v_k) \geq 2$ implies there is a vertex $v_{k+1} \in V(T) \setminus \{v_{k-1}\}$ with $v_kv_{k+1} \in E(G)$; define $W_{k+1} = (v_0, v_1, \cdots, v_n)$ and continue this process until $k+1 = n$. Now $v_i = v_j$ for some $0 \leq i < j \leq n$ by the Pigeonhole Principle. Choosing a pair $0 \leq i < j \leq n$ such that $v_i = v_j$ and $v_k \neq v_m$ for all $i \leq k < m < j$, we see $(v_i, v_{i+1}, \cdots, v_{j-1}, v_j)$ is a cycle in T; but this is absurd, as T is acyclic. Hence T must have a leaf. □

We leave the following corollary as an exercise for students to test their understanding.

Corollary. *Let G be a graph with $n \in \mathbb{Z}^+$ vertices. The following are equivalent.*

1. *Graph G is a tree.*
2. *Graph G has $n-1$ edges and is connected.*
3. *Graph G has $n-1$ edges and is acyclic.*