Notes on Graph Coloring

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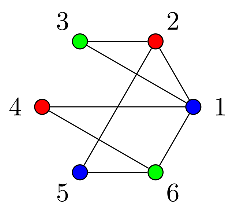
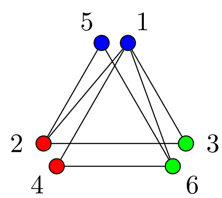
Definition. A *k-coloring* of a (simple) graph G is a function $c: V(G) \to [k]$ such that for all $u, v \in V(G)$ we have $c(v) = c(u)$ implies $uv \notin E(G)$. A graph is *k-colorable* when it has a k-coloring.

Typically we give pictures wherein vertices are actually colored.

Example 1. The following is a 3-coloring of a graph.



Remark. A k-coloring of a graph amounts to a separation of its vertices in k parts so that within parts no two vertices are connected by an edge. We move vertices with the same color from our above example into their respective parts to illustrate this idea below.



Proposition. *Every k-colorable graph is $(k+1)$-colorable.*

Proof. Let G be a graph and assume G has a k-coloring $c: V(G) \to [k]$. Let $c': V(G) \to [k+1]$ by the formula $c'(v) = c(v)$. Let $u, v \in V(G)$ be arbitrary. If $c'(u) = c'(v)$, then $c(u) = c'(u) = c'(v) = c(v)$; as c is a coloring of G, we see $uv \notin E(G)$. Hence c' is a $(k+1)$-coloring of G. □

Given a graph G, what is the minimum number of colors necessary to produce a coloring of G?

Definition. The *chromatic number* of (simple) graph G is

$$\chi(G) := \min \{k \in \mathbb{N} : G \text{ has a } k\text{-coloring}\}.$$

Remark. To prove $\chi(G) = k$ we must

1. color G with k colors, and
2. prove no coloring uses fewer colors.

Example 2. For the following classes of graphs, we can determine the chromatic number completely!

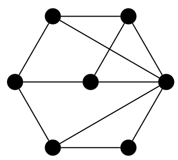
1. $\chi(K_n) = n$.

2. $\chi(P_n) = \min\{2, n\}$.

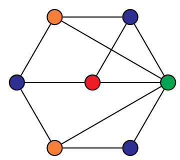
3. $\chi(C_n) = \begin{cases} 2 & \text{if } n \text{ is even} \\ 3 & \text{if } n \text{ is odd} \end{cases}$.

Exercise. Provide proofs for each of the above examples.

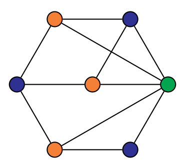
Example 3. Consider the graph G given below.

$G =$ 

We give a 4-coloring of G below.



Thus we have shown $\chi(G) \leq 4$. We can improve on this by turning the red vertex orange.



Hence we have that $\chi(G) \leq 3$ by our above coloring. Moreover $K_3 \leq G$, so every coloring of G yields a coloring of K_3. Now $\chi(G) < 3$ implies $\chi(K_3) < 3$, which is absurd; thus $\chi(G) \geq 3$. Hence $3 \leq \chi(G) \leq 3$, and we conclude $\chi(G) = 3$.

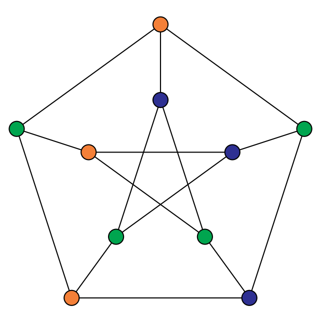
This example illustrates an important technique; we can prove lower bounds on $\chi(G)$ by using subgraphs and simple computations of those subgraphs. We collect this observation in the following proposition.

Proposition. *Let G be a graph. For every subgraph $H \leq G$ we have $\chi(H) \leq \chi(G)$.*

Idea of Proof. Every coloring of G yields a coloring of H. □

We end by coloring our favorite example: the Petersen graph!

Example 4. We exhibit a 3-coloring of the Petersen graph P below.



Thus $\chi(P) \leq 3$. Moreover, $C_5 \leq P$, so $3 = \chi(c_5) \leq \chi(P) \leq 3$. Hence $\chi(P) = 3$.