Notes on Eulerian and Hamiltonian Graphs

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All graphs here are assumed to be finite. We begin by extending a definition from a previous lecture.

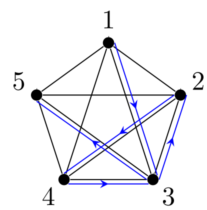
Definition. Let G be a graph.

A *walk* in G is a sequence $W = (v_0, e_1, v_1, \cdots, v_{n-1}, e_n, v_n)$ such that for all $i \in [n]$ we have e_i is an edge of G with ends v_{i-1} and v_i. Walk W is *closed* when $v_0 = v_n$; otherwise W is *open*.

Two vertices x and y are *connected* in G when there is a walk from x to y in G.

A *trail* in G is a walk which does not repeat any edges.

Example 1. The walk $W = (1, 13, 3, 23, 2, 24, 4, 34, 3, 35, 5)$ is a trail in the graph K_5.



Note that even though a trail cannot reuse edges, it may repeat vertices.

Definition. An *Euler trail* in graph G is a trail which uses every edge of G.

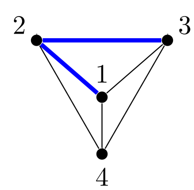
Showing a graph has an Euler trail amounts to exhibiting such a trail.

Example 2. Every cycle graph C_n has an Euler trail, obtained by traveling around the cycle.

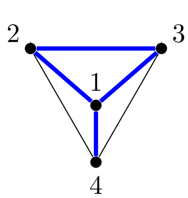
Example 3. The complete graph K_5 has an Euler trail, namely

$$W = (1, 12, 2, 23, 3, 34, 4, 45, 5, 15, 1, 13, 3, 35, 5, 25, 2, 24, 4, 14, 1).$$

Example 4. The graph K_4 does not have an Euler trail. First note that K_4 is a simple graph, so every walk in K_4 is determined by its sequence of vertices. Assume to the contrary that $W = (v_0, v_1, v_2, v_3, v_4, v_5, v_6)$ determines an Euler trail in K_4. Permuting labels we may assume $W = (1, 2, 3, v_3, v_4, v_5, 1)$; note $v_3 \in \{1, 4\}$.



If $v_3 = 1$, then $v_4 = 4$ as 12 and 13 are already in $W = (1, 2, 3, 1, 4, v_5, 1)$; but now $1v_5$ is already used by W. Thus $v_3 = 4$; if $v_4 = 2$, we cannot walk further—thus $W = (1, 2, 3, 4, 1, 3, v_5, 1)$, and we can't walk along 24.



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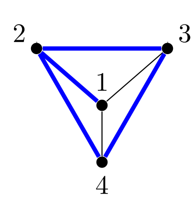

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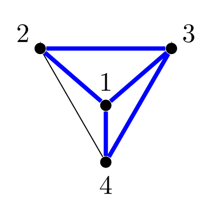
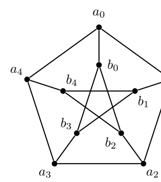


Hence K_4 does not have any Euler trail.

The above argument is somewhat painful, and required us to analyze all possible walks in K_4 (with some minor simplifications). In general, we would like a criterion to decide whether or not a graph G has an Euler trail by some simpler means. For example, does the Petersen graph have an Euler trail?



We will find such a criterion by analyzing two cases: when G has a closed/open Euler trail.

Proposition. *A connected graph has a closed Euler trail if and only if all of its vertices have even degree.*

Lemma. *If all vertices of graph G have degree at least two, then G has a cycle.*

Proof of Lemma. Let G be a graph having all vertices of degree at least two, and let $v_0 \in V(G)$. As $\deg(v_0) \geq 2$ there is an edge incident to v_0. Indeed we may walk from v_0 to a vertex v_1 by some edge e_1; let $W_1 := (v_0, e_1, v_1)$. Having $W_k = (v_e, e_1, v_1, e_2, ..., v_{k-1}, e_k, v_k)$, either $v_k = v_i$ for some $0 \leq i < k$ or not. If so, we have built a cycle $C = (v_i, e_{i+1}, v_{i+1}, \ldots, v_{k-1}, e_k, v_k)$. Otherwise, $\deg(v_k) \geq 2$ allows us to extend W_k to W_{k+1} by following an edge e_{k+1} from v_k to another vertex v_{k+1}. The walk $W_{\#V(G)}$ must repeat a vertex by the pigeonhole principle, so this procedure must yield a cycle. □

Proof of Proposition. Let G be a connected graph.

($\Longrightarrow$): Suppose G has a closed Euler trail W and let $x \in V(G)$ be arbitrary. Every instance of x in W is flanked by two incidences (unless $v_0 = x = v_n$, in which case we again have two incidences); as all incidences involving v appear exactly once we have that $\deg(v)$ is even.

($\Longleftarrow$): Assume every vertex of G has even degree. We proceed by (strong) induction on $\#E(G)$.

Base Case: If $\#E(G) = 0$, then $G = K_1$, and the unique walk in G is an Euler trail.

Inductive Step: Assume every connected graph with all vertices of even degree and having fewer edges than G has an Euler Trail. We may assume $\#E(G) > 0$. Obtain a cycle C in G by the Lemma. Let G' denote the graph obtained from G by removing all edges of C, and let $G_1, G_2, \ldots, G_k$ denote the connected components of G'. By our induction hypothesis, each G_i has an Euler trail W_i. Construct an Euler trail in G by writing C as a closed walk $C = (v_0, e_1, v_1, \ldots, v_{n-1}, e_n, v_n)$, for each $i \in [k]$ there is a smallest index $j_i \in [n]$ such that $v_{j_i} \in V(H_i)$. The desired Euler trail in G is given by following C, following W_i when encountering v_{j_i}, and then continuing along C again. □

We now leverage the above result to prove a similar result for graphs with an open Euler trail.

Corollary. *A connected graph has an open Euler trail if and only if it has exactly two vertices of odd degree.*

Proof. Let G be a connected graph. If G has an open Euler trail, then the first and last vertices of such a trail necessarily have odd degree and every other vertex has even degree. If G has exactly two vertices u and v of odd degree, we consider the graph G' obtained by adding an edge e between u and v. Every vertex of G' has even degree, so G' has a closed Euler trail W by the preceding proposition. Cyclically permuting W and exchanging the roles of u and v if necessary, we may assume $W = (u, e, v = v_0, e_1, v_1, \ldots, v_{n-1}, e_n, v_n = u)$. Thus the walk $W = (v = v_0, e_1, v_1, \ldots, v_{n-1}, e_n, v_n = u)$ is an open Euler trail in $G = G' \setminus e$. □

The Hamiltonian graphs are is a natural analogue of Eulerian graphs, replacing edges by vertices.

Definition. A *Hamilton cycle* is a cycle visiting every vertex exactly once.

Despite the strong parallelism between these questions, there is no known simple condition to characterizing Hamiltonicity; there are known separate sufficient conditions and necessary conditions.

Example 5. Consider the following classes of graphs.

1. The complete graphs K_n and the cycle graphs C_n Hamiltonian for all $n \geq 3$.
2. The path graph P_n is not Hamiltonian; more generally, any acyclic graph fails to be Hamiltonian.
3. The Petersen graph is not Hamiltonian (proving this requires some work).