Functions and Relations

1 Relations

Recall the definition of a relation.

Definition. Let A and B be sets. A *relation* $A \xrightarrow{R} B$ from A to B is a subset $R \subseteq A \times B$.

We will sometimes say R is a *relation on a set* S to mean a that R is a relation $S \xrightarrow{R} S$.
Here is a small example of a relation.

Example 1. We have a relation $\{1, 2, 3\} \xrightarrow{R} \{4, 5\}$ given by $R = \{(1, 4), (2, 4), (1, 5)\}$.

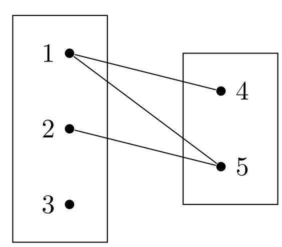
Relations are a mathematical model of relationships between the elements of various sets. The following is a very concrete example illustrating this idea.

Example 2. Let $P = \{x : x \text{ is a person}\}$. There are many meaningful relations on the set P.

- The relation $P \xrightarrow{sis} P$ is defined by $(x, y) \in sis$ when x and y are sisters.
- The relation $P \xrightarrow{mot} P$ is defined by $(x, y) \in mot$ when x is the mother of y.
- The relation $P \xrightarrow{stu} P$ is defined by $(x, y) \in stu$ when x was in a class taught by y.
- The relation $P \xrightarrow{fri} P$ is defined by $(x, y) \in fri$ when x any y are mutually friends.

Remark. It is cumbersome to write "$(x, y) \in R$". We often abbreviate using *infix notation* $x\ R\ y$ instead.

We will often depict relations using diagrams. For a relation $A \xrightarrow{R} B$, we will arrange the elements of A at the left, the elements of B at the right, and draw a line segment between two elements $a \in A$ and $b \in B$ when $a\ R\ b$. Doing so, we can depict the relation from Example 1 above in the following way:



Relations have very little structure; in particular, there are no requirements on the subset $R \subseteq A \times B$. If we add some simple conditions on our relations, they often become more meaningful.
The following notion is a mathematical abstraction of some fundamental properties of equality.

Definition. An *equivalence relation* on set S is a relation $R \subseteq S \times S$ such that

1. For all $x \in S$ we have $x\ R\ x$. (*Reflexive*)
2. For all $x, y \in S$ we have $x\ R\ y$ implies $y\ R\ x$. (*Symmetric*)
3. For all $x, y, z \in S$ we have both $x\ R\ y$ and $y\ R\ z$ implies $x\ R\ z$. (*Transitive*)

Notice that reflexivity, symmetry, and transitivity only make sense when we have a relation $R \subseteq S \times S$.

Example 3. The following are some examples of equivalence relations:

- Equality is an equivalence relation on any given set.
- Let P be the set of all people. The relation $P \xrightarrow{BDay} P$ defined by $x\ Bday\ y$ when x and y have the same birthday is an equivalence relation on P.

Example 4. The following set gives a relation on the set $S = \{0, 1, 2, 3, 4\}$:

$$\{(0, 0), (0, 1), (0, 2), (0, 3), (0, 4), (1, 1), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\}$$

Is this relation reflexive? Symmetric? Transitive?

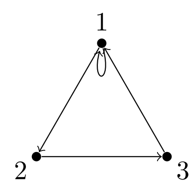
Problem 1. For each subset $X \subseteq \{\text{reflexive}, \text{symmetric}, \text{transitive}\}$ construct a relation which has precisely the properties in X. Find minimal examples (in terms of cardinality of the relation R and the set S).

Problem 2. Let $F \subseteq \operatorname{pow}(S)$ for set S, and suppose $\emptyset \notin F$.

1. Is the relation $F \xrightarrow{I} F$ where $(A, B) \in I$ when $A \cap B \neq \emptyset$ always an equivalence relation?
2. Is the relation $F \xrightarrow{D} F$ where $(A, B) \in D$ when $A \cap B = \emptyset$ always an equivalence relation?
3. Is the relation $F \xrightarrow{R} F$ where $(A, B) \in R$ when $\#A = \#B$ always an equivalence relation?

We can visualize a relation $R \subseteq S \times T$ is via a *directed graph* (we'll learn more about these later). Our directed graph has a point representing each element of $S \cup T$ and an arrow pointing from s to t whenever $s\ R\ t$.

Example 5. The relation $R = \{(1, 2), (2, 3), (3, 1), (1, 1)\}$ has the following directed graph:



Problem 3. Draw the directed graph for the relation from Example 4.

Another very important type of relation is called a partial ordering; this type of relation abstracts properties of the $\leq$ relation on real numbers.

Definition. A *partial order* on a set S is a reflexive and transitive relation R on S such that

1. For all $x, y \in S$ we have both $x\ R\ y$ and $y\ R\ x$ implies $x = y$. (*Antisymmetric*)

We have already seen some partial orders in the class. In particular, the following are partial orders.

1. Usual ordering on $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{N}_0$.
2. Divisibility Relation on $\mathbb{N}_0$.
3. The subset relation on $\operatorname{pow}(S)$ is a partial ordering.

2 Functions

Functions are the language of higher mathematics!

Definition. Let A and B be sets. A *function* $f\colon A \to B$ is a relation $f \subseteq A \times B$ such that for all $a \in A$ there is a unique $b \in B$ such that $(a, b) \in f$. The set A is called the *source* or *domain* of f, written $\operatorname{dom}(f) = A$. The set B is called the *target* or *codomain* of f, written $\operatorname{cod}(f) = B$.

Remark. Usually we will write $f(a) = b$ rather than $(a, b) \in f$ or $a\ f\ b$.

Example 6. For every set A there is an *identity function* $\operatorname{id}_A\colon A \to A$ having $\operatorname{id}_A(a) = a$ for all $a \in A$.

Functions f and g with the same domain and codomain are *equal* when $f(x) = g(x)$ for all $x \in \mathrm{dom}(f)$. As relations, functions are special; functions take an input and produce a unique output for that input. We can rephrase many of our previous results in terms of functions! Here is one example:

Proposition 1 (Pigeonhole Principle)**.** *Let $f\colon A \to B$ be a function with A and B finite sets. If $\#A > \#B$, then there are $a, a' \in A$ such that $f(a) = f(a')$.*

Given two compatible functions, we can get another function from them!

Definition. Functions $f\colon A \to B$ and $g\colon B \to C$ have *composition* $g \circ f\colon A \to C, \quad x \mapsto g(f(x))$.

Proposition 2. *For all $f\colon A \to B$, $g\colon B \to C$, and $h\colon C \to D$ we have $h \circ (g \circ f) = (h \circ g) \circ f$.*

Proof. For all $x \in \mathrm{dom}(f)$ we have the following equalities, completing the proof

$$(h \circ (g \circ f))(x) = h((g \circ f)(x)) = h(g(f(x))) = (h \circ g)(f(x)) = ((h \circ g) \circ f)(x). \qquad \square$$

Definition. Let $f : A \to B$ be a function.

1. The *preimage* of a set $S \subseteq B$ under f is the set $f^{-1}(S) = \{x \in A : f(x) \in S\}$.
2. The *image* of a set $T \subseteq A$ under f is the set $f(T) = \{f(x) \in B : x \in T\}$.

The next several propositions are straightforward applications of the definitions presented here. The proofs are left to you as a method of checking your understanding.

Proposition 3. *Let $f\colon A \to B$ be a function.*

1. *If $S \subseteq T \subseteq A$, then $f(S) \subseteq f(T)$.*
2. *If $S \subseteq T \subseteq B$, then $f^{-1}(S) \subseteq f^{-1}(T)$.*

Proposition 4. *Let $f\colon A \to B$ be a functon.*

1. *For all $S \subseteq A$ we have $S \subseteq f^{-1}(f(S))$.*
2. *For all $T \subseteq B$ we have $f(f^{-1}(T)) \subseteq T$.*

Proposition 5. *Let $f\colon A \to B$ be a function and $S, T \subseteq A$. The following all hold:*

1. $f(S \cup T) = f(S) \cup f(T)$
2. $f(S \cap T) \subseteq f(S) \cap f(T)$
3. $f(S \setminus T) \supseteq f(S) \setminus f(T)$

Problem 4. Find examples of functions and subsets for which the above subset relations are strict.

Proposition 6. *Let $f\colon A \to B$ be a function and $S, T \subseteq B$. The following all hold:*

1. $f^{-1}(S \cup T) = f^{-1}(S) \cup f^{-1}(T)$
2. $f^{-1}(S \cap T) = f^{-1}(S) \cap f^{-1}(T)$
3. $f^{-1}(S \setminus T) = f^{-1}(S) \setminus f^{-1}(T)$

Definition. Let $f\colon A \to B$ be a function.

1. Function f is *injective* or *into* when for all $a, a' \in A$ we have $f(a) = f(a')$ implies $a = a'$.
2. Function f is *surjective* or *onto* when for all $b \in B$ there exists an $a \in A$ such that $f(a) = b$.
3. Function f is *bijective* or a *one-to-one correspondence* when f is both injective and surjective.

Example 7. The identity function $\mathrm{id}_A\colon A \to A$ is bijective.

Problem 5. Write down examples of functions which are injective, surjective, and bijective. Can you write down a function which is injective but not surjective? How about one which is surjective but not injective?

Problem 6. If f is injective, can you strengthen Proposition 5? What if f is surjective?

In Calculus 2 you studied some inverse functions (the Inverse Function Theorem needs them!).

Definition. Let $f\colon A \to B$ be a function.

1. A *left inverse* of f is a function $g\colon B \to A$ such that $g \circ f = \mathrm{id}_A$.
2. A *right inverse* of f is a function $g\colon B \to A$ such that $f \circ g = \mathrm{id}_B$.
3. An *inverse* of f is a function $g\colon B \to A$ such that g is both a left inverse of f and a right inverse of f.

Example 8. The function id_A is its own inverse.

Problem 7. Find functions that have a left inverse but no right inverse and vice-versa.

The following proposition gives the relationship between invertibility and the properties above.

Proposition 7. *Let $f\colon A \to B$ be a function with $A \neq \emptyset$.*

1. *Function f has a left inverse if and only if f is injective.*
2. *Function f has a right inverse if and only if f is surjective.*
3. *Function f has an inverse if and only if f is bijective.*

Proof. Let $f\colon A \to B$ be a function.

Part 1: Supposing f has a left inverse g, then $(g \circ f)(a) = a$ for all $a \in A$. Thus $g(f(a)) = a$ for all $a \in A$. If $f(a) = f(a')$ for some $a, a' \in A$, then $a = g(f(a)) = g(f(a')) = a'$; hence f is injective. Supposing f is injective, fix an element $a_0 \in A$ (this is why we need $A \neq \emptyset$) and define

$$g(x) = \begin{cases} a & \text{if } x = f(a) \text{ for some } a \in A \\ a_0 & \text{otherwise} \end{cases}$$

for all $x \in B$. If $f(a) = f(a')$, then $a = a'$ by injectivity; thus g is well-defined. Moreover $(g \circ f)(x) = g(f(x)) = x$ for all $x \in A$; hence $g \circ f = \mathrm{id}_A$ and g is a left inverse of f.

Part 2: Supposing f has a right inverse g, then $(f \circ g)(b) = b$ for all $b \in B$. Thus for all $b \in B$ one has $g(b) \in A$ and $f(g(b)) = b$; hence f is surjective. Supposing f is surjective, we fix for all $b \in B$ an element $a_b \in A$ with $f(a_b) = b$.[1] Now define $g\colon B \to A$ by $g(b) = a_b$; note that this is well-defined by surjectivity of f. Moreover $(f \circ g)(x) = f(g(x)) = f(a_x) = x$ for all $x \in B$; hence $f \circ g = \mathrm{id}_B$ and g is a right inverse of f.

Part 3: Supposing f has an inverse, f has both a left and right inverse; hence by Part 1 and Part 2, f is both injective and surjective, and thus bijective. If f is bijective, then f is injective and surjective by definition; thus by Part 1 and Part 2 f has a left inverse g and a right inverse g'. Now

$$g = g \circ \mathrm{id}_B = g \circ (f \circ g') = (g \circ f) \circ g' = \mathrm{id}_A \circ g' = g'$$

and hence g is both a left and right invere for f. □

Proposition 8. *Let $f\colon A \to B$ be a function.*

1. *If f is injective, then for all $S \subseteq \mathrm{dom}(f)$ we have $f^{-1}(f(S)) = S$.*
2. *If f is surjective, then for all $T \subseteq \mathrm{cod}(f)$ we have $f(f^{-1}(S)) = S$.*

Proof. Exercise (HINT: you can use the preceeding proposition). □

[1] This is possible by an abstract axiom of set theory (called the Axiom of Choice). Mathematicians argued for a long time over whether or not this is a good axiom because it has a lot of weird consequences. To learn more, email me...