Math 324 Final Exam

1 July 2016

Name: Answer Key

1. Find the general solution to $2xy'' - y' + 2y = 0$.

$$y'' - \frac{1}{2x}y' + \frac{1}{x}y = 0$$

0 is not ordinary,

$$x\left(\frac{-1}{2x}\right) = \frac{-1}{2} \longrightarrow a_0 = \frac{-1}{2}$$

$$x^2\left(\frac{1}{x}\right) = x \longrightarrow b_0 = 0$$

$$q(r) = r(r-1) - \frac{1}{2}r = r^2 - \frac{3}{2}r = r\left(r - \frac{3}{2}\right)$$

roots $= 0, \frac{3}{2}$

$$\sum_{k=0}^{\infty} 2(k+r)(k+r-1)c_k x^{k+r-1} - \sum_{k=0}^{\infty}(k+r)c_k x^{k+r-1} + \sum_{k=0}^{\infty} 2c_k x^{k+r}$$

$k = j-1$
$j = k+1$

$$= \sum_{k=0}^{\infty}\left[2(k+r)(k+r-1) - (k+r)\right]c_k x^{k+r-1} + \sum_{k=1}^{\infty} 2c_{k-1}x^{k+r-1}$$

$$= \left[2r(r-1) - r\right]c_0 x^{r-1} + \sum_{k=1}^{\infty}\left[\left(2(k+r)(k+r-1) - (k+r)\right)c_k + 2c_{k-1}\right]x^{k+r-1}$$

$$2r^2 - 3r = 0 \qquad c_k = \frac{-2c_{k-1}}{(2k+2r-3)(k+r)}$$

$\underline{r=0}$

$$c_k = \frac{-2c_{k-1}}{(2k-3)k}$$

c_0 - any

$$c_1 = \frac{-2c_0}{-1} = 2c_0$$

$$c_2 = \frac{-2c_1}{2} = -c_1 = -2c_0$$

$$c_3 = \frac{-2c_2}{9} = \frac{4}{9}c_0$$

$\underline{r = \frac{3}{2}}$

$$c_k = \frac{-2c_{k-1}}{2k\left(k+\frac{3}{2}\right)}$$

c_0 - any

$$c_1 = \frac{-2c_0}{2\cdot\frac{5}{2}} = \frac{-2}{5}c_0$$

$$c_2 = \frac{-2c_1}{4\cdot\frac{7}{2}} = \frac{2}{5\cdot 7}c_0$$

$$c_3 = \frac{-2c_2}{6\cdot\frac{9}{2}} = \frac{-4}{5\cdot 7\cdot 27}$$

$$\boxed{y = c_1\left(1 + 2x - 2x^2 + \frac{4}{9}x^3 + \dots\right) + c_2\left(1 - \frac{2}{5}x + \frac{2}{35}x^2 - \frac{4}{945}x^3 + \dots\right)}$$

2. Find the general solution to $y'' - 2y' + 5y = e^x \sin(x)$.

$r^2 - 2r + 5 = 0$

$r = \frac{2 \pm \sqrt{4-20}}{2} = 1 \pm 2i$

$\boxed{y_c = c_1 e^x \cos(2x) + c_2 e^x \sin(2x)}$

$y_p = a_1 e^x \cos(x) + a_2 e^x \sin(x)$

$y_p' = a_1 e^x \cos(x) - a_1 e^x \sin(x)$
$+ a_2 e^x \sin(x) + a_2 e^x \cos(x)$

$= (a_1 + a_2) e^x \cos(x) + (a_2 - a_1) e^x \sin(x)$

$y_p'' = (a_1 + a_2) e^x \cos(x) - (a_1 + a_2) e^x \sin(x)$
$+ (a_2 - a_1) e^x \sin(x) + (a_2 - a_1) e^x \cos(x)$

$= 2a_2 e^x \cos(x) - 2a_1 e^x \sin(x)$

$S = 2a_2 C - 2a_1 S \quad -2(a_1 + a_2) C \quad -2(a_2 - a_1) S + 5a_1 C + 5a_2 S$

$= 3a_1 C + 3a_2 S$

$a_1 = 0 \quad a_2 = \frac{1}{3}$

$\boxed{y_p = \frac{1}{3} e^x \sin(x)}$

$y = y_p + y_c$

3. Find the general solution to $y'' + 3y' + 2y = \sin(e^x)$.

$$r^2 + 3r + 2 = 0$$
$$\shortparallel$$
$$(r+2)(r+1)$$

roots $= -2, -1$

$$\boxed{y_c = c_1 e^{-2x} + c_2 e^{-x}}$$

$$W = \begin{vmatrix} e^{-2x} & e^{-x} \\ -2e^{-2x} & -e^{-x} \end{vmatrix} = -e^{-3x} + 2e^{-3x} = e^{-3x}$$

$$W_1 = \begin{vmatrix} 0 & e^{-x} \\ \sin(e^x) & \cdots \end{vmatrix} = -e^{-x}\sin(e^x)$$

$$W_2 = \begin{vmatrix} e^{-2x} & 0 \\ \cdots & \sin(e^x) \end{vmatrix} = e^{-2x}\sin(e^x)$$

$r = z \rightarrow dr = dz$
$ds = \sin(z) \rightarrow s = -\cos(z)$

$$u_1' = -e^{2x}\sin(e^x) \rightarrow u_1 = -\int z\sin(z)\,dz = -\left(-z\cos(z) + \sin(z)\right) = z\cos(z) - \sin(z)$$
$$= e^x\cos(e^x) - \sin(e^x)$$

$$u_2' = e^x\sin(e^x) \rightarrow u_2 = -\cos(e^x)$$

$$y_p = u_1y_1 + u_2y_2 = e^{-x}\cos(e^x) - e^{-2x}\sin(e^x) - e^{-x}\cos(e^x) = \boxed{-e^{-2x}\sin(e^x)}$$

$$y = y_p + y_c$$

4. Solve the initial value problem: $t^2y'' - 5ty' + 8y = 8t^6, y(\frac{1}{2}) = 0, y'(\frac{1}{2}) = 0$.

$r(r-1) - 5r + 8 = 0$

$\shortparallel$

$r^2 - 6r + 8$

$\shortparallel$

$(r-4)(r-2)$

$y_c = c_1 x^4 + c_2 x^2$

$W = \begin{vmatrix} x^4 & x^2 \\ 4x^3 & 2x \end{vmatrix} = 2x^5 - 4x^5 = -2x^5$

$W_1 = \begin{vmatrix} 0 & x^2 \\ 8x^4 & \cdots \end{vmatrix} = -8x^6$

$W_2 = \begin{vmatrix} x^4 & 0 \\ \cdots & 8x^4 \end{vmatrix} = 8x^8$

$u_1' = +4x \rightarrow u_1 = 2x^2$

$u_2' = -4x^3 \rightarrow u_2 = -x^4$

$y_p = 2x^6 - x^6 = x^6$

$y = x^6 + c_1 x^4 + c_2 x^2$

$0 = y(\tfrac{1}{2}) = \frac{1}{64} + c_1 \frac{1}{16} + c_2 \frac{1}{4}$

$y' = 6x^5 + 4c_1 x^3 + 2c_2 x$

$0 = y'(\tfrac{1}{2}) = \frac{6}{32} + c_1 \frac{1}{2} + c_2$

$$\boxed{y = x^6 - \frac{1}{2} x^4 + \frac{1}{16} x^2}$$

$2 + 4c_1 = 0$

$c_1 = -\frac{1}{2}$

$1 + 4c_1 + 16c_2 = 0 \qquad 8 + 32c_1 + 128c_2 = 0$

$12 + 32c_1 + 64c_2 = 0$

$4 - 64c_2 = 0$

$\frac{4}{64} = c_2 = \frac{1}{16}$

5. Solve the initial value problem: $y' + y = f(t), y(0) = 0$ where $f(t) = \begin{cases} 1 \text{ when } 0 \le t < 1 \\ -1 \text{ when } t \ge 1 \end{cases}$ $= 1 - \mathcal{U}(t-1) - \mathcal{U}(t-1)$

$= 1 - 2\mathcal{U}(t-1)$

$sY(s) - y(0) + Y(s) = \frac{1}{s} - \frac{2}{s}e^{-s}$

$Y(s) = \frac{1}{s(s+1)} - \frac{2}{s(s+1)}e^{-s} = \frac{1}{s} - \frac{1}{s+1} - \frac{2}{s}e^{-s} + \frac{2}{s+1}e^{-s}$

$\frac{1}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1} = \frac{1}{s} - \frac{1}{s+1}$

$1 = As + Bs + A$

$A = 1 \quad B = -1$

$$\boxed{y(t) = 1 - e^{-t} - 2\mathcal{U}(t-1) + 2e^{-(t-1)}\mathcal{U}(t-1)}$$

6. Find the general solution to $y' = \frac{(x+1)^2 - 2y}{x^2-1}$.

$$\frac{2}{x^2-1} = \frac{A}{x-1} + \frac{B}{x+1} = \frac{1}{x-1} - \frac{1}{x+1}$$

$$y' + \frac{2}{x^2-1} y = \frac{(x+1)^2}{x^2-1}$$

$$2 = Ax + A + Bx - B$$

$$e^{\int \frac{2}{x^2-1} dx} = e^{\ln(x-1) - \ln(x+1)}$$

$$A - B = 2 \rightarrow 2A = 2 \rightarrow A = 1$$

$$A + B = 0 \rightarrow B = -1$$

$$= \frac{x-1}{x+1}$$

$$\left(\left(\frac{x-1}{x+1}\right)y\right)' = \frac{(x+1)^2}{(x-1)(x+1)} \cdot \frac{(x-1)}{(x+1)} = 1$$

$$\left(\frac{x-1}{x+1}\right)y = x + c$$

$$\boxed{y = \frac{x^2+x}{x-1} + \frac{cx+c}{x-1} = \frac{x^2 + (c+1)x + c}{x-1}}$$

7. Find the general solution to $y' = \frac{-x}{x^2y+4y}$.

$M = x \qquad M_y = 0$

$N = x^2y + 4y \qquad N_x = 2xy$ not exact!

$\dfrac{M_y - N_x}{N} = \dfrac{-2xy}{x^2y+4y}$ not a function of x!

$\dfrac{N_x - M_y}{M} = \dfrac{2xy}{x} = 2y$ is a function of y!

$\mu = e^{\int 2y\,dy} = e^{y^2}$

$\tilde{M} = xe^{y^2} \qquad \tilde{M}_y = 2xye^{y^2}$ exact!

$\tilde{N} = x^2ye^{y^2} + 4ye^{y^2} \qquad \tilde{N}_x = 2xye^{y^2}$

$f = \int xe^{y^2}dx = \frac{1}{2}x^2e^{y^2} + g(y)$

$= \int x^2ye^{y^2} + 4ye^{y^2}dy = \frac{1}{2}x^2e^{y^2} + 2e^{y^2} + h(x)$

$\frac{1}{2}x^2e^{y^2} + 2e^{y^2} = c$

$e^{y^2} = \dfrac{c}{\frac{1}{2}x^2+2}$

$y^2 = \ln\left(\dfrac{c}{\frac{1}{2}x^2+2}\right)$

$\boxed{y = \pm\sqrt{\ln\left(\dfrac{c}{\frac{1}{2}x^2+2}\right)}}$

8. Find the general solution to $y' = \sin(5x)$.

$$\boxed{y = -\frac{1}{5}\cos(5x) + c}$$

(Bonus) Name the five Mother Sauces of French cuisine.

1. Bechamel or White sauce
2. Velouté sauce
3. Espagnole sauce
4. Hollandaise sauce
5 Tomato sauce