

# Math 324 Exam 2

24 June 2016

Name: Answer key

1. For each, determine if 0 is an ordinary point, a regular singular point, or an irregular singular point. If 0 is a regular singular point, then find the indicial polynomial and compute its roots.

(a)  $4t^2y'' - 4te^t y' + 3\cos(t)y = 0$

$$y'' - \frac{e^t}{t} y' + \frac{3}{4} \frac{\cos(t)}{t^2} = 0$$

not ordinary

$$t\left(-\frac{e^t}{t}\right) = -e^t \rightarrow a_0 = -1$$

$$t^2\left(\frac{3}{4} \frac{\cos(t)}{t^2}\right) = \frac{3}{4} \cos(t) \rightarrow b_0 = \frac{3}{4}$$

(b)  $t^2y'' + \sin(t)y' + \cos(t)y = 0$

$$y'' + \frac{\sin(t)}{t^2} + \frac{\cos(t)}{t^2} = 0$$

not ordinary

$$t\left(\frac{\sin(t)}{t^2}\right) = \frac{\sin(t)}{t} = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2k}}{(2k+1)!} \rightarrow a_0 = 1$$

$$t^2\left(\frac{\cos(t)}{t^2}\right) = \cos(t) \rightarrow b_0 = 1$$

(c)  $t^2y'' - 5y' + 3t^2y = 0$

$$y'' - \frac{5}{t^2} y' + 3y = 0$$

not ordinary

$$t\left(-\frac{5}{t^2}\right) = -\frac{5}{t} \text{ not analytic at } 0$$

regular singular

$$q(r) = r(r-1) - r + \frac{3}{4}$$

$$r^2 - 2r + \frac{3}{4}$$

$$r = \frac{2 \pm \sqrt{4-3}}{2} \\ = \frac{3}{2}, \frac{1}{2}$$

regular singular

$$q(r) = r(r-1) + r + 1$$

$$r^2 + 1$$

$$\text{roots} = \pm i$$

irregular singular

(d)  $t^2y'' - t^3y' + t^2y = 0$

$$y'' - ty' + y = 0$$

$$P(x) = -t \quad Q(x) = 1 \quad \text{both analytic at } 0$$

ordinary

2. Find the general solution to  $t^2y'' - 5ty' + 9y = t^3$ .

Cauchy-Euler

$$q(r) = r(r-1) - 5r + 9$$

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$$r^2 - 6r + 9$$

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$$(r-3)(r-3)$$

roots = 3, 3

$$y_c = c_1 t^3 + c_2 t^3 \ln(t)$$

Variation of parameters

$$y'' - \frac{5}{t}y' + \frac{9}{t^2}y = t$$

$$W = \begin{vmatrix} t^3 & t^3 \ln(t) \\ 3t^2 & 3t^2 \ln(t) + t^2 \end{vmatrix} = 3t^5 \ln(t) + t^5 - 3t^5 \ln(t) = t^5$$

$$W_1 = \begin{vmatrix} 0 & t^3 \ln(t) \\ t & \dots \end{vmatrix} = -t^4 \ln(t)$$

$$W_2 = \begin{vmatrix} t^3 & 0 \\ 3t^2 & t \end{vmatrix} = t^4$$

$$u_1' = -\frac{\ln(t)}{t} \rightarrow u_1 = \frac{-(\ln(t))^2}{2}$$

$$u_2' = \frac{1}{t} \rightarrow u_2 = \ln(t)$$

$$y_p = -\frac{1}{2}t^3(\ln(t))^2 + t^3(\ln(t))^2 = \frac{1}{2}t^3(\ln(t))^2$$

$$y = y_p + y_c$$

3. Given that  $y_1 = e^{t^2}$  is a solution to  $y'' - 4ty' + (4t^2 - 2)y = 0$ , find the general solution.

$$\begin{aligned}y_2 &= ue^{t^2} \\y_2' &= u'e^{t^2} + 2tue^{t^2} = (u' + 2tu)e^{t^2} \\y_2'' &= u''e^{t^2} + 2tu'e^{t^2} + u'2te^{t^2} + u2e^{t^2} + u^4t^2e^{t^2} \\&= (u'' + 4tu' + (4t^2+2)u)e^{t^2}\end{aligned}$$

plug in  $y_2$ :

$$0 = (u'' + 4tu' + (4t^2+2)u)e^{t^2} + (-4tu' - 8t^2u)e^{t^2} + (4t^2u - 2u)e^{t^2}$$

$$= u''$$

$$u' = 1$$

$$u = t$$

$$y_2 = te^{t^2}$$

$$y = c_1y_1 + c_2y_2$$

4. Solve the initial value problem:  $y''' + 2y'' - y' - 2y = \sin(3t)$ ,  $y(0) = 0$ ,  $y'(0) = 0$ ,  $y''(0) = 1$ .

$$s^3 Y(s) - s^2 y(0) - sy'(0) - y''(0) + 2s^2 Y(s) - 2sy(0) - 2y'(0) \\ - sY(s) + y(0) - 2Y(s) = \frac{3}{s^2+9}$$

$$(s^3 + 2s^2 - s - 2)Y(s) = \frac{3}{s^2+9} + 1$$

$$\begin{aligned} & \frac{s^2+3s+2}{s-1} \quad Y(s) = \frac{3}{(s^2+9)(s-1)(s+2)(s+1)} + \frac{1}{(s-1)(s+2)(s+1)} \\ & \frac{3s^2-s-2}{-(s^3-s^2)} \\ & \frac{-3s^2-3s}{2s-2} \\ & \frac{s^2+12}{(s^2+9)(s-1)(s+2)(s+1)} \end{aligned}$$

$$\frac{s^2+12}{(s^2+9)(s-1)(s+2)(s+1)} = \frac{As+B}{s^2+9} + \frac{C}{s-1} + \frac{D}{s+2} + \frac{E}{s+1}$$

$$s^2+12 = As(s-1)(s+2)(s+1) + B(s-1)(s+2)(s+1) + C(s^2+9)(s+2)(s+1) \\ + D(s^2+9)(s-1)(s+1) + E(s^2+9)(s-1)(s+2)$$

you received full credit if you got this far.