

Math 324 Exam 1

10 June 2016

Name: Answer key

- Find the general solution to $y'' + y = x^2$.

$$r^2 + 1 = 0$$

$$\text{roots} = \pm i$$

$$y_c = c_1 \cos(x) + c_2 \sin(x)$$

D^3 annihilates x^2

$$\text{so } y_p = a_1 + a_2 x + a_3 x^2$$

$$y_p' = a_2 + 2a_3 x$$

$$y_p'' = 2a_3$$

$$2a_3 + a_1 + a_2 x + a_3 x^2 = x^2$$

$$\text{so } a_3 = 1$$

$$a_2 = 0$$

$$2a_3 + a_1 = 0 \rightarrow a_1 = -2$$

$$y_p = x^2 - 2$$

$$y = y_p + y_c$$

2. Find the general solution to $y' + e^x y = 3e^x$.

$$\text{integrating factor} = e^{\int e^x dx} = e^{e^x}$$

$$(e^{e^x} y)' = 3e^x e^{e^x}$$

$$e^{e^x} y = 3 \int e^x e^{e^x} dx = 3 \int e^w dw = 3e^{e^x} + C$$

$$\begin{aligned} w &= e^x \\ dw &= e^x dx \end{aligned}$$

$$y = 3 + ce^{-e^x}$$

3. Solve the initial value problem: $y''' - 9y'' + 27y' - 27y = 0$, $y(0) = 1$, $y'(0) = 2$, $y''(0) = 3$.

$$0 = r^3 - 9r^2 + 27r - 27 \quad (r-3)(r^2 - 6r + 9) = (r-3)^3$$

$$\begin{array}{r} 3 \text{ is a root, so} \\ \text{divide by } r-3 \\ \hline r-3 \left[\begin{array}{r} r^2 - 6r + 9 \\ r^3 - 9r^2 + 27r - 27 \\ -(r^3 - 3r^2) \\ -6r^2 + 27r - 27 \\ -(-6r^2 + 18r) \\ 9r - 27 \end{array} \right] \end{array}$$

roots = 3, 3, 3

$$y = c_1 e^{3x} + c_2 x e^{3x} + c_3 x^2 e^{3x}$$

$$\begin{aligned} y' &= 3c_1 e^{3x} + 3c_2 x e^{3x} + c_2 e^{3x} \\ &\quad + 3c_3 x^2 e^{3x} + 2c_3 x e^{3x} \end{aligned}$$

$$\begin{aligned} y'' &= \underline{9c_1 e^{3x}} + \underline{3c_2 e^{3x}} + \underline{9c_2 x e^{3x}} \\ &\quad + \underline{3c_2 e^{3x}} + \underline{9c_3 x^2 e^{3x}} + \underline{6c_3 x e^{3x}} \\ &\quad + \underline{2c_3 e^{3x}} + \underline{6c_3 x e^{3x}} \end{aligned}$$

$$1 \approx y(0) = c_1 \rightarrow \boxed{c_1 = 1}$$

$$2 = y'(0) = 3c_1 + c_2 \rightarrow \boxed{c_2 = -1}$$

$$3 = y''(0) = 9c_1 + 6c_2 + 2c_3 \rightarrow \boxed{c_3 = 0}$$

$$y = e^{3x} - x e^{3x}$$

4. Find the general solution to $y'' + y = \sec^2(x)$.

$$y_c = c_1 \cos(x) + c_2 \sin(x)$$

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix} = 1$$

$$W_1 = \begin{vmatrix} 0 & \sin(x) \\ \sec^2(x) & \cos(x) \end{vmatrix} = -\sin(x) \sec^2(x) = -\sec(x) \tan(x)$$

$$W_2 = \begin{vmatrix} \cos(x) & 0 \\ -\sin(x) & \sec^2(x) \end{vmatrix} = \cos(x) \sec^2(x) = \sec(x)$$

$$u_1' = -\sec(x) \tan(x) \rightarrow u_1 = -\sec(x)$$

$$u_2' = \sec(x) \rightarrow u_2 = \int \sec(x) dx = \ln(\sec(x) + \tan(x))$$

$$y_p = -\sec(x) \cos(x) + \ln(\sec(x) + \tan(x)) \sin(x)$$

$$y = y_c + y_p$$

5. (Bonus) Find the general solution to $y'' + y = x^2 + \sec^2(x)$.

$$y = x^2 - 3 + \ln(\sec(x) + \tan(x)) \sin(x) + c_1 \cos(x) + c_2 \sin(x)$$