Curve Sketching Notes

Gathering Information

Domain: The set of all real numbers for which the function is defined; should be written as a union of intervals.

y-intercept: The point at which the graph of the function f intersects (or intercepts) the y-axis; explicitly, the point $(0, f(0))$. If 0 is not in the domain of f, then clearly f has no y-intercept.

x-intercept(s) [or roots]: A list of all points at which the graph of the function f intersects (or intercepts) the x-axis; explicitly, the list of all points whose x-coordinate is a solution to the equation $f(x) = 0$.

End behavior: Check what happens with the function values $f(x)$ as goes to ∞ and as x goes to $-\infty$, if it makes sense ('as x approaches ∞' makes sense if (a, ∞) is in the domain for some number a, while 'as x approaches $-\infty$' makes sense if $(-\infty, b)$ is in the domain for some number b).

Asymptote(s):

(1) **Horizontal:** Refer back to the previous entry, end behavior. If $\lim_{x\to\infty} f(x) = L$, then the graph of f has a horizontal asymptote to the right, the line $y = L$. If $\lim_{x\to-\infty} f(x) = M$, then the graph of f has a horizontal asymptote to the left, the line $y = M$.

(2) **Vertical:** Let V denote the set of all points that are not in the domain of f but are near the domain of f; that is, points v such that $f(v)$ does not exist but $f(x)$ exists for all x in an interval of the form (a, v) [we say f is defined to the left of v] or of the form (v, b) [we say f is defined to the right of v]. If f is defined to the left of v, find $\lim_{x\to v^-} f(x)$, and if f is defined to the right of v, find $\lim_{x\to v^+} f(x)$. If either limit is ∞ or $-\infty$, the graph of f has a vertical asymptote the line $x = v$.

Critical Point(s): A list of points $(c, f(c))$ where either $f'(c)$ does not exist or $f'(c) = 0$. The number c is called a critical value for the function f.

Local Max/Min: Let c be a critical value of f and let a, b be real numbers such that c is the only critical value of f in the interval (a, b).

(1) **First Derivative Test:** The function f has a local max at the point $(c, f(c))$ if and only if $f' > 0$ on (a, c) and $f' < 0$ on (c, b), and has a local minimum at the point $(c, f(c))$ if and only if $f' < 0$ on (a, c) and $f' > 0$ on (c, b).

(2) **Second Derivative Test** If $f'(c)$ does not exist, use the First Derivative Test. Suppose $f'(c) = 0$. If $f''(c) = 0$, use the First Derivative Test. Otherwise, f has a local max at the point $(c, f(c))$ if $f''(c) < 0$ and f has a local min at the point $(c, f(c))$ if $f''(c) > 0$.

Intervals of Increasing/Decreasing: The answer here will be two lists of open intervals; the intervals on which f is increasing and the intervals on which f is decreasing. Let C denote the set of all critical values of f, V the set of points mention in the vertical asymptote section, and let S be the set of all points which are in C or in V. Let u and v be in S with $u < v$ and such that nothing in S is in the interval (u, v). The function f is increasing on the interval (u, v) if and only if $f' > 0$ on (u, v), and is decreasing on the interval (u, v) if and only if $f' < 0$ on (u, v).

Concavity: The answer here will be two lists of open intervals; the intervals on which f is concave up (slope increasing) and the intervals on which f is concave down (slope decreasing). Let D denote the set of critical values of f', B the set of critical values c of f for which $f'(c)$ does not exist, and V as above. Let T be the set of all values which are in D or B or V. Let u and v be in T with $u < v$ and such that nothing in T is in the interval (u, v). The function f is concave up on the interval (u, v) if and only if f' is increasing on (u, v) if and only if $f'' > 0$ on (u, v), and is concave down on the interval (u, v) if and only if f' is decreasing on the interval (u, v) if and only if $f'' < 0$ on (u, v).

Inflection Point(s): A list of points $(c, f(c))$ where the concavity to the left of c is different from the concavity to the right of c; the graph of f has an inflection point at $(c, f(c))$ if there are numbers $a < c < b$ such that f is concave up on the interval (a, c) and concave down on the interval (c, b) or if f is concave down on the interval (a, c) and is concave up on the interval (c, b).

Sketching:
Once all the parts above are down, we just have to sketch our graph. Mark and label all intercepts, local max/min, and inflection points. Use dashed lines to draw the asymptotes and label each one with its equation; that is, $x = v$ for a vertical asymptote, or $y = L$ for a horizontal asymptote. If f has a horizontal asymptote on just one side, be sure to show that by only drawing it on the correct side. Finally, sketch the graph piece by piece according to the increasing/decreasing and concavity information along with the asymptote behavior, connecting pieces together continuously (if they actually do connect); if there is a point where f is defined but f' is not, indicate that (could be with a sharp turn like the tip of a letter 'v', but be sure to follow the concavity information as well as increasing/decreasing information).