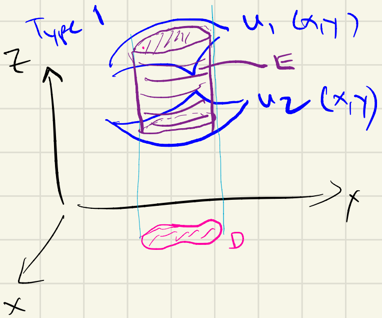
# 16.9: Divergence Theorem

**Def:** the solid region E is simple if it is simultaneously of type 1, 2, 3.




**The Divergence Theorem:** Let E be a simple solid region and let S be the boundary surfac of E given w/ positive (outward) orientation.

Let $\vec{F}$ be a vector field w/ comp. fnc's that have cts partial derivatives on an open region containing E. Then,

$$\boxed{\iint_S \vec{F} \cdot d\vec{S}} = \boxed{\iiint_E \operatorname{div} \vec{F} \, dV}$$

flux of $\vec{F}$ across S = the triple integral of div $\vec{F}$ over the region bounded by S ex) Find the flux of $\vec{F}(x,y,z) = \langle z, y, x \rangle$ over the unit sphere $x^2+y^2+z^2=1$ (call it $S$).

Sol: By the div. thm,

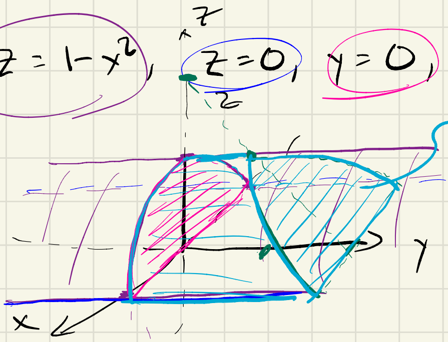
$$\iint_S \vec{F}\cdot d\vec{S} = \iiint_E \operatorname{div}(\vec{F})\, dV$$

$E: \quad x^2+y^2+z^2 \le 1 \qquad \therefore \iint_S \vec{F}\cdot d\vec{S} = \iiint_{x^2+y^2+z^2\le 1} 1\, dV = \boxed{\frac{4\pi}{3}}$

$\operatorname{div} \vec{F} = 1$

ex) Evaluate $\iint_S \vec{F}\cdot d\vec{S}$ where $\vec{F} = \langle xy,\ y^2+e^{xz^2},\ \sin(xy) \rangle$ and $S$ is the boundary of the region bounded by $z = 1-x^2$, $z=0$, $y=0$, $y+z=2$.

Sol:




$E:\quad 0 \le y \le 2-z$

$0 \le z \le 1-x^2$

$-1 \le x \le 1$

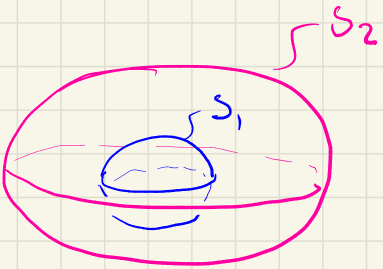
$$\iint_S \vec{F}\cdot d\vec{S} = \iiint_E \operatorname{div}(\vec{F})\, dV$$

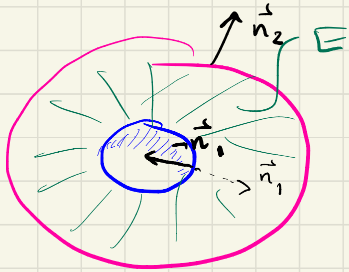
$$= \int_{-1}^{1}\int_0^{1-x^2}\int_0^{2-z} y + 2y + 0 \ dV$$

$$= \int_{-1}^{1}\int_0^{1-x^2}\int_0^{2-z} 3y \ dV$$

$$= \cdots = 184/35.$$

It turns out, the divergence theorem can be used in the following setting:




Suppose $E_1$ is the region bounded by $S_1$,

and $E_2$ ——————〃—————— $S_2$.

Note ( $E_1$ is contained in $E_2$ and $E = E_2 - E_1$. )

Then,
$$\iiint_E \operatorname{div}\vec{F}\,dV = \iiint_{E_2} \operatorname{div}\vec{F}\,dV - \iiint_{E_1} \operatorname{div}\vec{F}\,dV$$

$$\overset{\text{div. thm.}}{=} \iint_{S_2} \vec{F}\cdot d\vec{S} - \iint_{S_1} \vec{F}\cdot d\vec{S}$$

$$= \iint_{S_2} \vec{F}\cdot\vec{n}_2\,dS + \iint_{S_1} \vec{F}\cdot(-\vec{n}_1)\,dS$$