Counting Problems and Generating Functions

Example 1: Find the number of solutions of

$$y_1 + y_2 + y_3 = 17,$$

where $2 \leq y_1 \leq 5, 3 \leq y_2 \leq 6$, and $4 \leq y_3 \leq 7$.

Proof.

The number of solutions is the coefficient of x^{17} in the expansion of :

$$(x^2 + x^3 + x^4 + x^5) \cdot (x^3 + x^4 + x^5 + x^6) \cdot (x^4 + x^5 + x^6 + x^7)$$

where the exponents of x in the first factor correspond to the possible values of y_1, exponents of x in the second factor corresponds to possible values of y_2 and similarly for y_3.

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where the exponents of x in the first factor correspond to the possible values of y_1, exponents of x in the second factor corresponds to possible values of y_2 and similarly for y_3. A solution like $y_1 = 4, y_2 = 6, y_3 = 7$ correspond to $x^4 \cdot x^6 \cdot x^7 = x^{17}$ □

Exercise 1: Solve using generating functions: In how many ways can 10 identical books be distributed among 3 children , so that each child receives at least 2 but not more than 4 books.

Example 2: Use generating functions to determine the number of ways to insert tokens worth $\$1, \$2,$ and $5 into a vending machine to pay for an item that costs r dollars when:

Case: 1 The order in which the tokens are inserted does not matter

Proof.

The generating function is given by:

$$(1+x+x^2+x^3+\cdots)(1+x^2+x^{2(2)}+x^{2(3)}+\cdots)(1+x^5+x^{5(2)}+x^{5(3)}+\cdots)$$

The first powers of x in the first factor tells you the number of 1 tokens and powers of x in the second factor tells number of $2 tokens. So we find the coefficient of x^r □

Case 2: The order in which the tokens are inserted matters (to pay \$3, inserting \$1 followed by \$2 is different from inserting \$2 followed by \$1.)

Proof.

The number of ways to insert n tokens to get a total of r dollars is the coefficient of r in $(x + x^2 + x^5)^n$

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Proof.

The number of ways to insert n tokens to get a total of r dollars is the coefficient of r in $(x + x^2 + x^5)^n$ As any number of tokens can be inserted to produce r dollars, the generating function is:

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Proof.

The number of ways to insert n tokens to get a total of r dollars is the coefficient of r in $(x + x^2 + x^5)^n$ As any number of tokens can be inserted to produce r dollars, the generating function is:

$$1 + (x + x + x^5) + (x + x^2 + x^5) + (x + x^2 + x^5)^2 + \cdots = \frac{1}{1 - (x + x^2 + x^5)}$$

□

Exercises

Use generating function to describe how to solve the following:

- The number of ways to select 14 balls from a jar containing 100 red balls, 100 blue balls, and 100 green balls so that no fewer than 3 and no more than 10 blue balls are selected. Assume that the order in which the balls are selected does not matter.

Exercises

Use generating function to describe how to solve the following:

- The number of ways to select 14 balls from a jar containing 100 red balls, 100 blue balls, and 100 green balls so that no fewer than 3 and no more than 10 blue balls are selected. Assume that the order in which the balls are selected does not matter.
- Given a combinatorial interpretation of the coefficient of x^6 in the expansion $(1 + x + x^2 + x^3 + \cdots)^n$
- Obtain a generating formula for the number of ways that the sum n can be obtained when a six-faced is rolled repeatedly and the order of the rolls matters.

Section 9: Relations

Definition: Let A and B be two sets. A binary relation from A to B is a subset of $A \times B$. **Recall:** $A \times B = \{(a, b) : a \in A, b \in B\}$. (**Question**: If $|A| = n, |B| = m$, what is $|A \times B|$?)

Example: Let $A = \{0, 1, 2\}$ and $B = \{a, b\}$. Then $R = \{(0, a), (1, b), (2, b), (0, b)\}$ is an example of a relation.

In the definition of a relation R, if $A = B$, we say R is a relation on A

Question: How many relations are there of A if $|A| = n$

Properties of Relations on A

- A relation R is called reflexive if $(a, a) \in R$ for every $a \in A$ (Examples: The relation " $\leq$ " is reflexive on the set of natural numbers)
- R is called symmetric if $(b, a) \in R$ whenever $(a, b) \in R$ (Example : The relation)
- R is called anti-symmetric if whenever (a, b) and (b, a) are R,then $a = b$
- R is called transitive if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$

More Exercises

- How many reflexive relations are there on A if $|A| = n$
- Check whether the following relations on set of all integers satisfies any of the above properties: The relations defined by $(x, y) \in R$ if and only if :
  (i) $x + y = 0$
  (ii) $x \equiv y \pmod 7$
  (iii) $xy \geq 1$
  (iv) x is a multiple of y