

# Permutations with Indistinguishable Objects

**Example 1:** How many different strings can be made by reordering the letters of the word "SUCCESS"

Proof.

If all the letters are distinct, then there are  $7!$  of arrangements. But there are 3 indistinguishable  $S$  so we divide by  $3!$  (for example  $S$  in position 1, 3, 5 is the same ordering with  $S$  in 5, 3, 1 remember there are  $3!$  of such ordering). Also there are 2  $C$ 's. Hence the number of arrangements is  $\frac{7!}{3!2!}$   $\square$

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The number of different permutations of  $n$  objects where there are  $n_1$  indistinguishable of type 1,  $n_2$  indistinguishable of type 2,  $\dots$ ,  $n_k$  indistinguishable of type  $k$  is:

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## Exercises:

- i There are 5 questions on a discrete math exam. How many ways are there to assign scores to the problems if the sum of the scores is 30 and each question is worth at least 2 points
- ii How many ways are there to distribute six indistinguishable balls into nine distinguishable boxes.
- iii How many ways are there to distribute 12 distinguishable objects into six distinguishable boxes
- iv How many ways are there to travel in  $x, y, z, w$  from the origin  $(0, 0, 0, 0)$  to the point  $(4, 3, 5, 4)$  by taking steps one unit in the positive  $x$ , positive  $y$ , positive  $z$  or positive  $w$ . (Hint: Try a 2 dimensional version as example)
- v How many ways are there to distribute 15 distinguishable objects into five distinguishable boxes so that the boxes have 1, 2, 3, 4, 5 objects in them respectively.

## Section 8.4: Generating Functions

### Definitions:

- ▶ A finite series is of the form :  $\sum_{k=0}^n a_k = a_0 + a_1 + a_2 + \cdots + a_n$
- ▶ An infinite series is of the form:  
$$\sum_{k=0}^{\infty} a_k = a_0 + a_1 + a_2 + \cdots + a_k + a_{k+1} + \cdots$$
- ▶ A power series  $P(x)$  is a series of the form :  $P(x) = \sum_{k=0}^n a_k x^k$

### Examples:

- ▶  $1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n}$  is an example of a finite series
- ▶  $1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^k} + \cdots$  is an infinite series.
- ▶  $1 + x + x^2 + x^3 + \cdots + x^k + \cdots$  is an example of a power series.

# Generating Function

**Definition:** A generating function for the sequence  $a_0, a_1, a_2, \dots, a_k, \dots$  of real numbers is the infinite series:

$$G(x) = a_0 + a_1x + a_2x^2 + \dots + a_kx^k + \dots = \sum_{k=0}^{\infty} a_kx^k$$

**Examples:** Find the generating functions for the following sequence of numbers:

The sequence 1, 1, 1, 1, 1, 1.

**Proof.**

$G(x) = 1 + x + x^2 + x^3 + x^4 + x^5 = \frac{(x^6-1)}{(x-1)}$  (recall sum of terms in a geometric sequence?) □

The sequence  $1, a, a^2, a^3, \dots,$

**Proof.**

$$G(x) = 1 + ax + a^2x^2 + a^3x^3 + \dots = \frac{1}{1-ax} \quad (\text{Note, this is for } |x| < \frac{1}{|a|}) \quad \square$$

## More Exercises:

Find a closed form formula for the generating function  $G(x)$  for the following sequence of numbers:

- i  $2, 2, 2, 2, 2, \dots, 2$  (this is 2 written  $n$  times)
- ii  $0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, \dots$
- iii  $\binom{7}{0}, \binom{7}{1}, \binom{7}{2}, \binom{7}{3}, \dots, \binom{7}{7}, 0, 0, 0, \dots$