Theorem

A simple graph is bipartite if and only if it is possible to assign one of two different colors to each vertex of the graph so that no two adjacent vertices are assigned the same color.

Proof.

Suppose that $G = (V, E)$ is a bipartite simple graph. Then $V = V_1 \cup V_2$, assign one color to each vertex in V_1 and a second color to each vertex in V_2. No two adjacent vertices are assigned the same color.
Conversely, Suppose the graph G is 2-colored . Let V_1 be set of vertices with one color and V_2 the set of vertices with the other color. $\square$

Bipartite Graph and Matchings

Bipartite graphs can be used to model applications such as Job Assignments. Here is an example to illustrate: **Example:**Suppose there are m employees in a group and n different jobs that need to be done, where $m \geq n$. Each employee is trained to do one or more of these n jobs. We would like to assign an employee to each job. We can construct a bipartite graph G as follows:

- V_1 denotes the employees
- V_2 denotes the jobs
- There is an edge from an employee to a job he/she has be trained to to do.

Scenario 1: There four employees : Alvarez, Brady, Chen and Davis, and there are the following jobs: Requirements, Architecture, Implementation and Testing. Suppose that Alvarez has been trained to do requirements and testing, Brady has been trained in architecture, implementation and testing, Chen has been trained to do requirements, architecture and implementation and Davis has only been trained to do requirements. Represent the model using a bipartite graph.
The **goal** is to assign an employee to each job, so that every job every job has an employee assigned to it, and no employee is assigned to more than one job. A solution is (Alvarez - Testing, Brady-Implementation, Chen -Architecture, Davis-Requirements)

Scenario 2 Suppose in a second group, there are Peter, Kwame, Yuan, Tom; and suppose that the same four jobs needs to be completed. Suppose Peter is trained to do architecture, Kwame is trained to do requirements and implementation, Yuan is trained to do architecture and Tom is trained to do requirements, architecture and testing. Draw a bipartite graph of the problem
Question: Is there a way to assign a job to each person so that no one get more than one job?

Definition A matching M in a simple graph $G = (V, E)$ is a subset of the edges so that no two edges are incident to the same vertex.

- If a vertex is the endpoint of an edge in M it is said to be matched, otherwise unmatched.
- A maximum matching is a matching with the largest number of edges
- A matching in a bipartite graph $G = (V_1, V_2, E)$ is said to be a complete matching if $|M| = |V_1|$ (that is each vertex in V_1 is matched to exactly a vertex in V_2)

Necessary and Sufficient Conditions for Complete Matchings

We will determine when a complete matching from V_1 to V_2 exists in a bipartite graph $G = (V_1, V_2, E)$ **Definition:**Let $G = (V, E)$ be a graph and $v \in V$. Then the set of all neighbors of the vertex v denoted by $N(v)$ is the neighborhood of v. If $A \subseteq V$, then $N(A) = \bigcup_{v \in A} N(v)$.

Theorem

HALL's MARRIAGE THEOREM: The bipartite graph $G = (V_1, V_2, E)$ has a complete matching from V_1 to V_2 if and only if $|N(A)| \geq |A|$ for all subsets A of V_1 (**).

Proof.

Suppose there is a complete matching M from V_1 to V_2. Then if $A \subseteq V$, there for any vertex $v \in A$, there is an edge connecting v to a distinct vertex in V_2. Hence $|N(A)| \geq |A|$. □

Proof

Proof.

Now, suppose $|N(A)| \geq |A|$ for all $A \subseteq V_1$. We will proceed by strong induction on the size of $|V_1|$.

Basis step: If $|V_1| = 1$. Then V_1 contains a single vertex v_0. As $|N(v_0)| \geq 1$, there is atleast one edge connecting v_0 and a vertex $w_0 \in V_2$. Any such edge forms a complete matching from V_1 to V_2.

Inductive hypothesis: Let k be a positive integer. If $G = (V, E)$ is a bipartite graph (V_1, V_2, E) and $|V_1| = j \leq k$, then there is a complete matching M from V_1 to V_2 whenever the condition that $|N(A)| \geq |A|$ for all $A \subseteq V_1$ is satisfied.

Now suppose that $H = (W, F)$ is a bipartite graph (W_1, W_2, F) with $|W_1| = k + 1$ □

Case 1: Suppose that for all integers j with $1 \leq j \leq k$, the vertices in every subset of j elements from W_1 are adjacent to at least $j+1$ elements of W_2. Then we select a vertex $v \in W_1$ and an element $w \in N(v)$. We now consider the bipartite graph $H' = (W_1 - \{v\}, W_2 - \{w\})$. Now $|W_1 - \{v\}| = k$ and condition $(**)$ is still satisfied. Hence H' has a complete matching by induction.

Case 2: Suppose that for some j with $1 \leq j \leq k$, there is a subset W_1' of j vertices such that there are exactly j neighbors denoted by W_2'. We consider the bipartite graph $(W_1 - W_1', W_2 - W_2')$. (Why does this bipartite graph still satisfies condition (**)?). By strong induction, $(W_1 - W_1', W_2 - W_2')$ has a complete matching (what do we do to get a complete matching for (W_1, W_2)?)

Exercises

Section 10.2 Exercises: $27, 29, 31, 33$