



# The inverse Kakeya problem

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## Abstract

We prove that the largest convex shape that can be placed inside a given convex shape  $Q \subset \mathbb{R}^d$  in any desired orientation is the largest inscribed ball of  $Q$ . The statement is true both when “largest” means “largest volume” and when it means “largest surface area”. The ball is the unique solution, except when maximizing the perimeter in the two-dimensional case.

**Keywords** Kakeya problem · Convexity · Turning · Brunn-Minkowski

## 1 Introduction

The well-known Kakeya problem, originally asked by Soichi Kakeya in 1917, is the following question: What is the minimum area region  $Q$  in the plane in which a needle of length 1 can be turned through  $360^\circ$  continuously, and return to its initial position [6]? When  $Q$  is required to be convex, the answer is the equilateral triangle of height one [9]. For general  $Q$ , however, Besicovitch showed that a region  $Q$  of arbitrarily small measure exists [2,3]. Kakeya-type problems have received considerable attention in the literature, as there are strong connections to problems in number theory [4], geometric combinatorics [12], arithmetic combinatorics [8], oscillatory integrals, and the analysis of dispersive and wave equations [11].

Being able to rotate a needle through  $360^\circ$  inside  $Q$  clearly implies that it can be placed in  $Q$  in any desired orientation. Bae et al. [1] showed that the converse holds more generally for convex shapes in the plane: If a planar convex compact shape  $P$  can be placed in a planar convex compact shape  $Q$  in any desired orientation, then  $P$  can also be rotated through  $360^\circ$  inside  $Q$ . A natural generalization of Kakeya’s problem is therefore to ask, given a planar convex compact shape  $P$ , what is the minimum area convex shape  $Q$  such that  $P$  can be

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placed in  $Q$  in any desired orientation. This problem still seems to be wide open, the answer is not even known when  $P$  is an equilateral triangle or a square.

In this short note, we consider the inverse of this question. We are given a convex compact shape  $Q \subset \mathbb{R}^d$ ,  $d \geq 2$ , and we ask: what is the largest shape  $P$  that can be placed in  $Q$  in any desired orientation?

We show that, independently of the shape of  $Q$ , the answer is always a spherical ball, and therefore  $P$  is the largest inscribed ball of  $Q$ . The result is true both for maximizing the volume of  $P$  and for maximizing the surface area of  $P$ . The answer is always unique, *except* when maximizing the perimeter of  $P$  in the planar case. For instance, inside a unit square both a unit-diameter disk and a unit-diameter Reuleaux-triangle can be turned. Both have the same perimeter, but the disk has larger area.

The result for maximizing the volume is a consequence of the well-known Brunn–Minkowski theorem. For the surface-area result, we make use of the generalized Brunn–Minkowski theorem, a theorem that deserves to be better known. This proof does not cover the planar case, so we give an elementary proof based on Minkowski sums.

Our characterization also solves the computational question of computing the largest convex  $P$  that can be placed in any desired orientation in a given convex polyhedron  $Q \subset \mathbb{R}^d$ , since the largest inscribed ball can be computed efficiently as a linear program.

For completeness, let us observe that the case  $d = 1$  is simple and has the same behavior as  $d = 2$ . Indeed, in  $\mathbb{R}$  the convex shapes are segments, there is a unique longest segment that can be placed inside a segment  $Q$  in both orientations, namely  $Q$  itself, and all segments of positive length inside  $Q$  have a boundary of the same size, namely two points.

On the other hand, if we ask for the  $P$  that maximizes the *diameter*, then the answer is different: it is a line segment whose length is the smallest width of  $Q$ . In general, this is longer than the diameter of the largest inscribed ball, for instance when  $Q$  is an equilateral triangle.

## 2 Minkowski sums

For two convex shapes  $P$  and  $Q$  in  $\mathbb{R}^d$ , the Minkowski sum  $P + Q$  is the set  $\{p + q \mid p \in P, q \in Q\}$ . Minkowski sums allow us to interpolate between two convex shapes  $P_0$  and  $P_1$ : for  $0 \leq \lambda \leq 1$ , we define  $P_\lambda := (1 - \lambda)P_0 + \lambda P_1$ .

**Lemma 2.1** *Let  $Q$  be a convex shape in  $\mathbb{R}^d$ , let  $P_0, P_1 \subseteq Q$  be convex shapes, and let  $0 \leq \lambda \leq 1$ . Then  $P_\lambda \subseteq Q$ .*

**Proof** Let  $p \in P_\lambda$ . Then  $p = (1 - \lambda)p_0 + \lambda p_1$ , with  $p_0 \in P_0 \subseteq Q$  and  $p_1 \in P_1 \subseteq Q$ . Since  $p_0, p_1 \in Q$  and  $Q$  is convex,  $p \in Q$ .

For a convex shape  $Q \subset \mathbb{R}^d$ , we introduce  $\mathfrak{R}(Q)$  as the family of all convex shapes  $P \subset \mathbb{R}^d$  that can be placed in  $Q$  in any desired orientation. Lemma 2.1 immediately implies the following:

**Lemma 2.2** *Let  $Q$  be a convex shape in  $\mathbb{R}^d$ , let  $P_0, P_1 \in \mathfrak{R}(Q)$ , and let  $0 \leq \lambda \leq 1$ . Then  $P_\lambda \in \mathfrak{R}(Q)$ .*

**Proof** Consider an arbitrary rotation  $\rho$ . We need to argue that there exists a translation  $t$  such that  $\rho P_\lambda + t \subset Q$ . By assumption, there are translations  $t_0$  and  $t_1$  such that  $\rho P_0 + t_0 \subset Q$  and  $\rho P_1 + t_1 \subset Q$ . Setting  $t = (1 - \lambda)t_0 + \lambda t_1$ , we have

$$\rho P_\lambda + t = \rho((1 - \lambda)P_0 + \lambda P_1) + (1 - \lambda)t_0 + \lambda t_1 = (1 - \lambda)(\rho P_0 + t_0) + \lambda(\rho P_1 + t_1).$$

Now Lemma 2.1 implies the claim.

We denote the  $d$ -dimensional volume of convex shape  $P$  by  $\psi_0(P) = \text{vol}(P)$ , and the  $(d - 1)$ -dimensional volume of the boundary of  $P$  by  $\psi_1(P) = \text{surf}(P)$ . The key ingredient for our proof is the following lemma:

**Lemma 2.3** *Let  $P_0, P_1 \subset \mathbb{R}^d$  be convex shapes with  $\psi_w(P_0) = \psi_w(P_1)$ , for  $w \in \{0, 1\}$  and  $d \geq 2 + w$ . Then  $\psi_w(P_{1/2}) \geq \psi_w(P_0)$ , and equality holds only when  $P_0$  and  $P_1$  are translates.*

For the volume case  $w = 0$ , Lemma 2.3 follows immediately from the well-known Brunn–Minkowski theorem [10, Theorem 6.1.1]. The general form follows from the generalized Brunn–Minkowski Theorem for mixed volumes. We need some notions from the theory of mixed volumes, see Busemann [5, Chapter 2] for an introduction.

Minkowski has shown that for  $r$  convex shapes  $K_1, \dots, K_r \subset \mathbb{R}^d$ , the volume of their linear combinations is a homogenous polynomial of degree  $d$ :

$$\text{vol}(\lambda_1 K_1 + \dots + \lambda_r K_r) = \sum_{i_1=1}^r \sum_{i_2=1}^r \dots \sum_{i_d=1}^r V(K_{i_1}, K_{i_2}, \dots, K_{i_d}) \lambda_{i_1} \lambda_{i_2} \dots \lambda_{i_d},$$

The coefficients  $V(K_{i_1}, \dots, K_{i_d})$  are called *mixed volumes*. Setting  $r = 1$  we see that  $V(K, \dots, K) = \text{vol}(K)$ . The mixed volumes for  $r = 2$  and  $K_2 = B$ , where  $B$  is the unit ball in  $\mathbb{R}^d$ , are known as *quermassintegrals*, and denoted

$$\begin{aligned} W_0(K) &= V(K, \dots, K), \\ W_1(K) &= V(K, \dots, K, B), \\ W_m(K) &= V(K[d - m], B[m]), \quad \text{for } m \in \{0, 1, \dots, d\} \end{aligned}$$

where the  $P[m]$  notation means that argument  $P$  is repeated  $m$  times.

The *generalized Brunn–Minkowski theorem* states that for  $m \in \{2, 3, \dots, d\}$  and convex shapes  $K_0, K_1, C_1, \dots, C_{d-m} \subset \mathbb{R}^d$ , the function

$$f(\lambda) := (V(K_\lambda[m], C_1, C_2, \dots, C_{d-m}))^{1/m}$$

is a concave function on the interval  $[0, 1]$ , where  $K_\lambda = (1 - \lambda)K_0 + \lambda K_1$ . (The Brunn–Minkowski theorem is the special case  $m = d$ .) Busemann [5, pp. 49–50] gave a short proof using the Aleksandrov–Fenchel inequality. When  $C_1, \dots, C_{d-m}$  are sufficiently smooth, then the function  $\lambda \mapsto f(\lambda)$  is a linear function only when  $K_0$  and  $K_1$  are homothets [10, Theorems 6.4.4 and 6.6.9]. This applies in particular to the quermassintegral case, and we obtain the following: For  $m \in \{2, 3, \dots, d\}$  and convex shapes  $K_0$  and  $K_1$ , the function

$$f(\lambda) := (W_{d-m}(K_\lambda))^{1/m}$$

is concave on the interval  $[0, 1]$ , and it is linear only when  $K_0$  and  $K_1$  are homothets. In particular, if  $W_{d-m}(K_0) = W_{d-m}(K_1)$ , then  $f(\lambda)$  is concave on the interval  $[0, 1]$ , and it is constant only when  $K_0$  and  $K_1$  are translates.

Lemma 2.3 follows from this by observing that  $\psi_0(K) = \text{vol}(K) = W_0(K)$  and  $\psi_1(K) = \text{surf}(K) = dW_1(K)$  [10, pg. 210].

Note that the proof does not include the case  $d = 2$  and  $w = 1$ . In fact, Lemma 2.3 does not hold in this setting, as  $\lambda \mapsto \text{peri}(P_\lambda)$  is a linear function on the interval  $[0, 1]$  for any convex  $P_0, P_1 \subset \mathbb{R}^2$ . We will return to this case in Sect. 4.

### 3 The main theorem when the optimum is unique

Our proof strategy is to consider an optimal shape  $P$  and argue that if  $P$  is not a spherical ball, then there is a shape  $P'$  with larger volume or surface area. For this argument to go through, it is therefore necessary to first argue that an optimal shape does indeed exist.

**Lemma 3.1** *Let  $Q$  be a given convex shape in  $\mathbb{R}^d$ , for  $d \geq 2$ , and let  $w \in \{0, 1\}$ . Then there exists  $R \in \mathfrak{K}(Q)$  such that for any  $P \in \mathfrak{K}(Q)$  we have  $\psi_w(P) \leq \psi_w(R)$ .*

**Proof** Let  $\omega = \sup_{P \in \mathfrak{K}(Q)} \psi_w(P)$ . For any  $i > 0$ , we can choose  $K_i \in \mathfrak{K}(Q)$  with  $\psi_w(K_i) > \omega - 1/i$  and such that the origin lies in  $K_i$ . This implies that all  $K_i$  are contained in a ball centered at the origin whose radius is the diameter of  $Q$ .

By Blaschke's selection theorem [7], there is a subsequence  $(K_{i_j})$  that converges in the Hausdorff sense to some compact convex shape  $K$ . For simplicity of presentation, we let  $(K_i)$  denote this converging subsequence.

By continuity of  $\psi_w$ , we have  $\psi_w(K) = \omega$ . To prove the lemma, it now suffices to prove that  $K \in \mathfrak{K}(Q)$ , that is, that  $K$  can be placed inside  $Q$  in any given orientation  $\rho$ .

We fix some rotation  $\rho$ . Since  $K_i \in \mathfrak{K}(Q)$ , there is a vector  $t_i \in \mathbb{R}^d$  such that  $\rho K_i + t_i \subset Q$ . Since the origin lies in  $K_i$ , we have  $t_i \in Q$ . Since  $Q$  is compact, this implies that the sequence  $(t_i)$  contains a converging subsequence. Let  $(t_i)$  again denote this subsequence, and denote  $t = \lim t_i$ .

Let  $a_i$  be the Hausdorff distance of  $K_i$  and  $K$ , and let  $b_i = |t_i - t|$ . It follows that the Hausdorff distance of  $\rho K_i + t_i$  and  $\rho K + t$  is at most  $a_i + b_i$ , which implies that  $\rho K_i + t_i$  converges in the Hausdorff sense to  $\rho K + t$ . Since  $\rho K_i + t_i \subseteq Q$  and  $Q$  is compact, this implies that  $\rho K + t \subseteq Q$ , so  $K$  can be placed in  $Q$  in orientation  $\rho$ .

We can now prove the main theorem:

**Theorem 3.1** *Let  $Q$  be a given convex shape in  $\mathbb{R}^d$ , for  $d \geq 2$ , let  $D$  be the largest spherical ball inscribed to  $Q$ , and let  $P \neq D$  be a convex shape that can be placed in  $Q$  in every orientation. Then  $\text{vol}(P) < \text{vol}(D)$ . If  $d \geq 3$ , then we also have  $\text{surf}(P) < \text{surf}(D)$ .*

**Proof** Let  $w \in \{0, 1\}$ . By Lemma 3.1, there exists  $P \in \mathfrak{K}(Q)$  that maximizes  $\psi_w(P)$ . If  $P$  is not a ball, then there is a rotation  $\rho$  such that  $P$  and  $\rho P$  are not homothets. But then Lemma 2.3 implies that  $\psi_w(\frac{1}{2}(P + \rho P)) > \psi_w(P)$ . On the other hand, since  $P, \rho P \in \mathfrak{K}(Q)$ , Lemma 2.2 implies that  $\frac{1}{2}(P + \rho P) \in \mathfrak{K}(Q)$ , a contradiction to the assumption that  $P$  maximized  $\psi_w(P)$ .

### 4 Largest perimeter

It remains to discuss the case of maximizing the perimeter in the plane. It is well known (and follows for instance from the Cauchy-Crofton formula) that  $\text{peri}(P + Q) = \text{peri}(P) + \text{peri}(Q)$  for any planar convex shapes  $P$  and  $Q$ .

We fix an integer  $\mu$ , and define  $\rho$  to be the rotation around the origin by angle  $\frac{2\pi}{\mu}$ . For a planar convex shape  $P$ , we define the  $\mu$ -average  $P^\mu$  of  $P$  to be the set

$$P^\mu := \frac{1}{\mu} \sum_{k=0}^{\mu-1} \rho^k P.$$

**Theorem 4.1** *Let  $P$  and  $Q$  be planar convex shapes such that  $\rho^k P$  can be translated into  $Q$  for every  $k \in \{0, 1, \dots, \mu - 1\}$ . Then  $P^\mu$  can be translated into  $Q$ , and we have  $\text{peri}(P^\mu) = \text{peri}(P)$ .*

**Proof** For  $i \in \{1, 2, \dots, \mu\}$  we define

$$P_i := \frac{1}{i} \sum_{k=0}^{i-1} \rho^k P.$$

We will prove by induction that  $P_i$  can be translated into  $Q$  and that  $\text{peri}(P_i) = \text{peri}(P)$ . Since  $P^\mu = P_\mu$ , this implies the theorem.

The base case is  $i = 1$ , where  $P_i = P_1 = P$ . Assume now that  $i \in \{2, \dots, \mu\}$  and that the statement holds for  $P_{i-1}$ . We observe that

$$P_i = \frac{1}{i} \rho^i P + \frac{i-1}{i} P_{i-1} = (1-\lambda) \rho^i P + \lambda P_{i-1} \quad \text{with } \lambda = 1 - \frac{1}{i}.$$

Since  $\rho^i P$  and  $P_{i-1}$  can be translated into  $Q$ , Lemma 2.1 implies that  $P_i$  can be translated into  $Q$ . We have  $\text{peri}(P_i) = (1-\lambda)\text{peri}(\rho^i P) + \lambda\text{peri}(P_{i-1}) = \text{peri}(P)$ .

Let now  $K_\mu$  denote a regular convex  $\mu$ -gon. We denote the edges of  $K_\mu$  in counter-clockwise order as  $e_1, e_2, \dots, e_\mu$ . A convex polygon  $P$  is called a  $\mu$ -polygon if every edge of  $P$  is parallel to an edge of  $K_\mu$ . We represent a  $\mu$ -polygon  $P$  as a vector  $\phi(P) = (a_1, a_2, \dots, a_\mu)$  in  $\mathbb{R}^\mu$ , where  $a_i$  is the length of the edge of  $P$  with the same outward normal as  $e_i$ . The vector  $\phi(P)$  represents  $P$  uniquely up to translations.

We observe the following:

- $\phi(\alpha P) = \alpha \phi(P)$  for  $\alpha > 0$ ;
- $\phi(P + R) = \phi(P) + \phi(R)$ ;
- If  $\phi(P) = (a_1, \dots, a_\mu)$  and  $\rho$  is a rotation with rotation angle  $\frac{2\pi}{\mu}$ , then  $\phi(\rho P) = (a_\mu, a_1, a_2, \dots, a_{\mu-1})$ .

As a consequence, the  $\mu$ -average  $P^\mu$  of  $P$  has vector

$$\phi(P^\mu) = (r, r, r, \dots, r) \quad \text{where } r = \frac{1}{\mu} \sum_{k=0}^{\mu-1} a_k = \frac{\text{peri}(P)}{\mu}.$$

In other words,  $P^\mu$  is a regular convex  $\mu$ -gon.

**Theorem 4.2** *Let  $Q$  be a given planar convex shape, let  $D$  be the largest disk inscribed to  $Q$ , and let  $P \in \mathfrak{K}(Q)$ . Then  $\text{peri}(P) \leq \text{peri}(D)$ .*

**Proof** Assume the claim is false, so that we have  $P \in \mathfrak{K}(Q)$  with  $\text{peri}(P) > \text{peri}(D)$ . Let  $\varepsilon := \text{peri}(P) - \text{peri}(D) > 0$ . If we choose  $\mu$  large enough, then there is a  $\mu$ -polygon  $R \subset P$  with  $\text{peri}(P) - \text{peri}(R) < \varepsilon/2$ . We consider its  $\mu$ -average  $R^\mu$ . By Theorem 4.1 we can translate  $R^\mu$  into  $Q$ , and have  $\text{peri}(R^\mu) = \text{peri}(R) \geq \text{peri}(P) - \varepsilon/2$ . Since  $R^\mu$  is a regular convex  $\mu$ -gon, we can ensure—by making  $\mu$  large enough—that there is a disk  $D' \subset R^\mu$  with  $\text{peri}(R^\mu) - \text{peri}(D') < \varepsilon/2$ . This implies that  $\text{peri}(P) - \text{peri}(D') < \varepsilon$ . Since  $D' \subset R^\mu \subset P$ , the disk  $D'$  cannot be larger than  $D$  and so we have  $\text{peri}(P) - \text{peri}(D) < \varepsilon$ , a contradiction to the choice of  $\varepsilon$ .

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