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Shadows of a Closed Curve

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A shadow of a geometric object A in a given direction v is the orthogonal projection of A on the hyperplane orthogonal to v. We show that any topological embedding of a circle into Euclidean d-space can have at most two shadows that are simple paths in linearly independent directions. The proof is topological and uses an analog of basic properties of degree of maps on a circle to relations on a circle. This extends a previous result that dealt with the case $d = 3$.

1 Introduction

Given a set A in $\mathbb{R}^d$, we define the i-th coordinate shadow of A as the image of A by the orthogonal projection to the coordinate hyperplane $\{(x_1, \ldots, x_d) \in \mathbb{R}^d : x_i = 0\}$. Recall that a simple closed curve is a topological embedding of a circle and a simple path is a topological embedding a segment. Suppose we want to draw a simple closed curve in $\mathbb{R}^d$ so as to maximize the number of shadows that are simple paths. It is easy to see that two shadows can be simple paths. Just consider the unit circle in a coordinate plane $A = \{(x_1, x_2, 0 \ldots, 0) \in \mathbb{R}^d : x_1^2 + x_2^2 = 1\}$. The 1st and 2nd coordinate shadows of A are simple paths, but all others are circles. We show that this is the best that can be done.

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Theorem 1.1. (version 1) A simple closed curve in $\mathbb{R}^d$ has at most two coordinate shadows that are simple paths.

By considering a curve up to linear transformations, Theorem 1.1 can be restated as follows:

Theorem 1.1. (version 2) For any simple closed curve γ in $\mathbb{R}^d$, it is not possible to project γ in three linearly independent directions such that the image by each projection is a simple path.

Coordinate shadows are a common and effective tool for visualizing and analyzing geometric objects in high-dimensional space. For example, orthogonal projections are used in classical methods for data compression [8, Chapter 4.26] and dimension reduction [7].

Trying to describe topological properties of a set A using topological properties of the coordinate shadows of A might seem futile at first glance, because so much information about the set is lost and also because coordinate shadows are geometric features that depend delicately on a choice of coordinates. Our result, however, alludes to a topological relation between a set and its coordinate shadows and provide an early step toward answering the following more general inquiry.

> Given an embedding of a topological space A in some Euclidean space of higher dimension, what does the topology of its shadows tell us about the topology of A?

This is in the spirit of tomography, which studies how a set A can be reconstructed from the volume of the intersection of A with lower dimensional spaces (the Radon transform of A). This question can be seen as an extreme case of sparse sampling in tomography where the information available is restricted to the support function of the Radon transform along lines in d linearly independent directions [3].

1.1 Background

This problem was motivated by the following question asked by H. W. Lenstra.

> Is there a simple closed curve in 3-space such that all three of its coordinate shadows are trees?

The original motivation for Lenstras question was Oskars puzzle cube, three mutually orthogonal rods that pass through slits in the sides of a hollow cube. The rods are joined at a common point, and the slits in the sides of the cube comprise three mazes. To move the rods to a desired configuration, all three of these mazes must be solved simultaneously. Lenstra originally asked if the three mazes could be designed so

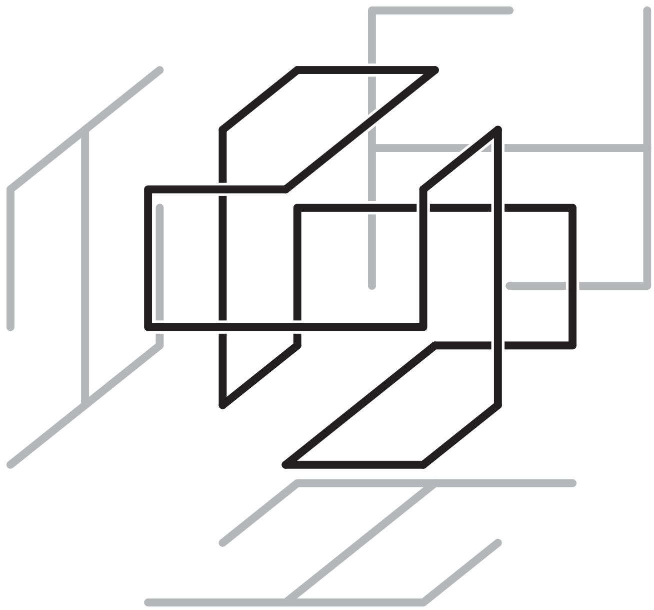

Fig. 1. A simple closed curve in 3-space such that all of its shadows are trees.

that the point where the three rods meet can move along a trajectory that returns to its starting position without backtracking, thus tracing a closed curve. Of course, none of the three mazes in the sides of the cube can contain a closed curve individually, since that would result in a side of the cube being disconnected.

An affirmative answer to this question was given by J. R. Rickard [9, p. 112] (see Figure 1) and several such curves were later shown to exist (e.g., [5]). More about the history of this problem can be found in the book *Mathematical Mind-Benders* by Peter Winkler [9, p. 118] under the name of Curve and Three Shadows. In fact, the cover of this book shows Rickard's curve.

A variant of this question is whether there is a simple closed curve in 3-space such that its three coordinate shadows are simple paths. This was asked in the Canadian Conference on Computational Geometry (CCCG) 2007 [2] and in the 2012 Mathematics Research Communities workshop. It has been shown in [1] that the answer is negative. On the other hand, the same paper showed that there does exist a simple path in 3-space such that all three of its shadows are simple closed curves [1]. Our result is a generalization of the question asked in CCCG 2007 to any d-space.

As part of the proof, we employ a method that was used by Goodman, Pach, and Yap to show that two mountain climbers starting at sea level on opposite sides of a mountain (in the plane) can climb the mountain so that both climbers are always at equal altitude while making their way to the top [4].

1.2 Organization and structure of proof

We will prove Theorem 1.1 by contradiction. Given a simple closed curve $\underline{\gamma}$ with three coordinate shadows that are each paths, we show in Section 2 that there cannot be a point $q_0 \in \underline{\gamma}$ that projects to an end point of each shadow. Then, in Section 4, we show that such a point q_0 must exist. To show that q_0 exists, we define a relation on the curve such that any fixed point of the relation would project to an end point of each shadow. Finally, we show that this relation does indeed have a fixed point. Before getting into the proof that q_0 must exist, we prove a special case of Theorem 1.1 in Section 3 to provide geometric intuition. For this, we use the fact that a degree -1 map from the circle to itself has a fixed point.

1.3 Notation and terminology

Let $\{x_i{=}c\}$ denote the hyperplane $\{(x_1, \ldots, x_d) \in \mathbb{R}^d : x_i = c\}$, and let $\mathrm{I} = [0, 1]$ denote the closed unit interval, $\mathbb{S}^1$ will be the circle, and $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ will be the torus. Let $e_1, \ldots, e_d$ denote the standard basis vectors in $\mathbb{R}^d$, and $e_1^*, \ldots, e_d^* : \mathbb{R}^d \to \mathbb{R}$ denote the dual coordinate functions. Let $\pi_i : \mathbb{R}^d \to \{x_i{=}0\}$ be the orthogonal projection to the i-th coordinate hyperplane. For a function $\varphi : X \to Y$, let $\underline{\varphi} = \varphi(X)$ denote the range of φ. Let X° denote the interior of a set.

Here we define a graph to be a one-dimensional cell complex. That is, a graph is a topological space consisting of a set of vertices and edges, where an edge between a pairs of vertices v, w is given by a homeomorphic copy of the unit interval with end points v, w, whose interior is disjoint from all other edges. Here it is enough to consider simple graphs, but allowing graphs to have loops or multiple edges in general will have no impact on our arguments. Generally we will be interested in graphs that are embedded in a torus where every vertex has degree 1 or 2.

Recall that the degree of a vertex in a graph is the number of edges incident to that vertex. We call this the graphical degree of a vertex to avoid confusion with the topological degree of a map, which will be discussed later.

2 End Points of Three Shadows

We will prove Theorem 1.1 by contradiction using the following lemma.

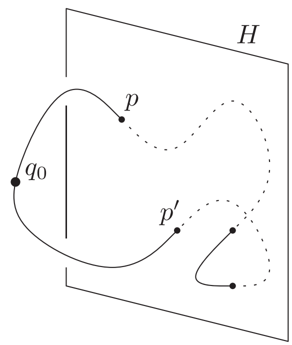

Fig. 2. The points p and p'.

Lemma 2.1. If γ is a simple closed curve in $\mathbb{R}^d$ with three coordinate shadows that are each a path, then there cannot be a point $q_0 \in \gamma$ that is projected to an end point of each of the three paths.

Proof. Suppose the lemma is false and let $\gamma : \mathbb{S}^1 \to \mathbb{R}^d$ be a simple closed curve with a point $q_0 \in \gamma$ such that $\pi_i(\gamma)$ is a path having $\pi_i(q_0)$ as an end point for $i = 1, 2, 3$. In fact, we will only need to use that q_0 projects to an end point of both the 2nd and 3rd coordinate shadows.

The curve γ cannot be contained in a hyperplane that is perpendicular to e_1; otherwise $\pi_1(\gamma)$ would be a translate of γ, but $\pi_1(\gamma)$ is a path. Therefore, there exists a hyperplane $H = \{x_1{=}c\}$ that intersects γ at multiple points but does not contain q_0. Let p and p' be the end points of the connected component of $\gamma \setminus H$ that contains q_0 (see Figure 2). Since H intersects γ in multiple points, $p \neq p'$. Let $\varphi : \mathrm{I} \to \gamma$ be a parameterization of γ that starts at q_0 and passes through p before p', and let $\varphi'(x) = \varphi(1-x)$ parameterize γ in the opposite direction. Also, let $\lambda_i : \mathrm{I} \to \pi_i(\gamma)$ be a parameterization of the i-th coordinate shadow that starts at the end point $\pi_i(q_0)$. The 1st coordinate of φ attains the value c for the 1st time at p,

$$p = \varphi\left(\inf\{t \in \mathrm{I} : \langle e_1, \varphi(t)\rangle = c\}\right).$$

Therefore, $\pi_2(p)$ is the point where $\pi_2 \circ \varphi$ attains the value c in the 1st coordinate for the 1st time, which is also the point where λ_2 attains the value c in the 1st coordinate for the 1st time. Similarly, φ' attains the value c in the 1st coordinate for the 1st time at p', so $\pi_2(p')$ must also be the point where λ_2 attains the value c in the 1st coordinate for the 1st time. Hence, $\pi_2(p) = \pi_2(p')$, which implies $p - p'$ is parallel to e_2. Likewise, since q_0

projects to an end point of both the 2nd and 3rd and coordinate shadows, $\pi_3(p) = \pi_3(p')$ by the same argument as for the 2nd coordinate shadow, so $p - p'$ is also parallel to e_3. Together these imply $p = p'$, which is a contradiction. ∎

3 Intuition

3.1 Fixed points and degree

This section briefly reviews the topological degree of a map and its relevant properties [6]. The topological degree $\deg(f)$ of a map $f : \mathbb{S}^1 \to \mathbb{S}^1$ is given by the number of times f wraps around the circle. That is, $\deg(f) = k$ when f can be continuously deformed to the map $f_k(u) = (\sin(k\theta_u), \cos(k\theta_u))$ where θ_u is the positive angle between the vector u and the vector $e_1 \in \mathbb{S}^1$. Topological degree may alternatively be defined in terms of induced maps on homology. For a simple closed curve $\gamma : \mathbb{S}^1 \to \mathbb{R}^d$ and a map $f : \underline{\gamma} \to \underline{\gamma}$, we let $\deg(f) = \deg(\gamma^{-1} \circ f \circ \gamma)$.

The two important properties of topological degree we use are as follows: the degree of a composition of maps is the product of their degrees $\deg(f \circ g) = \deg(f)\deg(g)$ and if $f : \mathbb{S}^1 \to \mathbb{S}^1$ does not have a fixed point then $\deg(f) = 1$. Briefly, if f does not have a fixed point, then

$$f_t(x) = \tilde{f}_t(x)/\|\tilde{f}_t(x)\|, \text{ where } \tilde{f}_t(x) = (1-t)f(x) - tx,$$

gives a continuous deformation from f to the antipodal map. The antipodal map wraps once around the circle in the positive direction and therefore has degree 1, so f also has degree 1.

In Section 4 we show how both of these properties can be generalized from maps on the circle to relations on the circle.

3.2 A special case

Before we give a complete proof of Theorem 1.1, we present the basic ideas involved. To do this we assume that the simple closed curve is embedded in $\mathbb{R}^d$ in a very particular way.

Theorem 3.1. (a special case of Theorem 1.1) A simple closed curve in $\mathbb{R}^d$ cannot have three coordinate shadows such that each shadow is a simple path and the projection map to each shadow is 2-to-1 on the interior of the path.

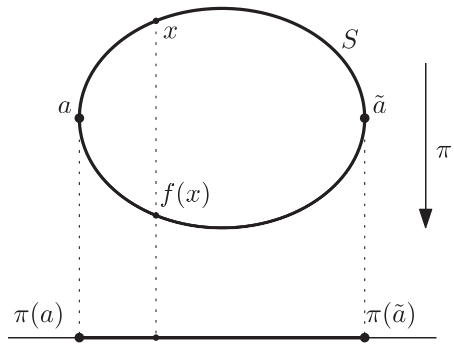

Fig. 3. The curve γ and the *flipping* function f.

Proof. Suppose that $\gamma : \mathbb{S}^1 \to \mathbb{R}^d$ is a simple closed curve such that $\pi_i(\gamma)$ is a path for $i = 1, 2, 3$ and that each π_i restricted to γ is a 2-to-1 map except for two points $a_i, \tilde{a}_i \in \gamma$ that are mapped to the boundary of its shadow.

We can define a function f_i that sends each point of γ where π_i is 2-to-1 to the other point with the same image, and let f_i be fixed elsewhere. That is, we have a continuous function $f_i : \gamma \to \gamma$ such that a_i and $\tilde{a}_i$ are its only fixed points and $\pi_i(x) = \pi_i(f(x))$ for every $x \in \gamma$ (see Figure 3). This function has topological degree -1 and essentially "flips" γ in the i-th direction. Consequently, the degree of the composition $f_3 \circ f_2 \circ f_1$ is -1 and therefore has a fixed point $q_0 \in \gamma$. Now consider the points

$$q_1 = f_1(q_0), \quad q_2 = f_2(q_1), \quad q_3 = f_3(q_2) = q_0.$$

The three vectors $q_i - q_{i-1}$ for $i = 1, 2, 3$ are respectively parallel to the 1st three elements of the canonical basis e_1, e_2, e_3 and add up to 0. Thus each of the vectors is $q_i - q_{i-1} = 0$, so $q_0 = q_1 = q_2$. This means that q_0 is a fixed point of each f_i and is therefore mapped to an end point of each of the paths, which is a contradiction by Lemma 2.1. ■

4 The General Case of Theorem 1.1

4.1 Fixed points of relations

For Theorem 3.1, the assumption that the projection map is 2-to-1 everywhere in the interior of the path allows us to define the function f_i that "flips" the closed curve. In general, many points may be projected to a single point on the shadow, so instead of a

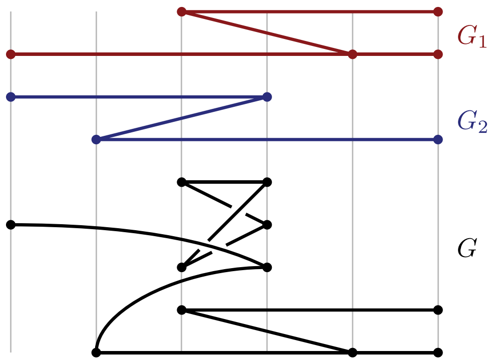

Fig. 4. The graphs G_1 and G_2 are given as planar drawing; the functions f_1 and f_2 are the projections in the vertical direction. Below is the fiber product $G = G_1 \times_{f_1,f_2} G_2$. In this representation the point $(x_1, x_2) \in G \subset G_1 \times G_2$ is vertically aligned with the points $x_1 \in G_1$ and $x_2 \in G_2$.

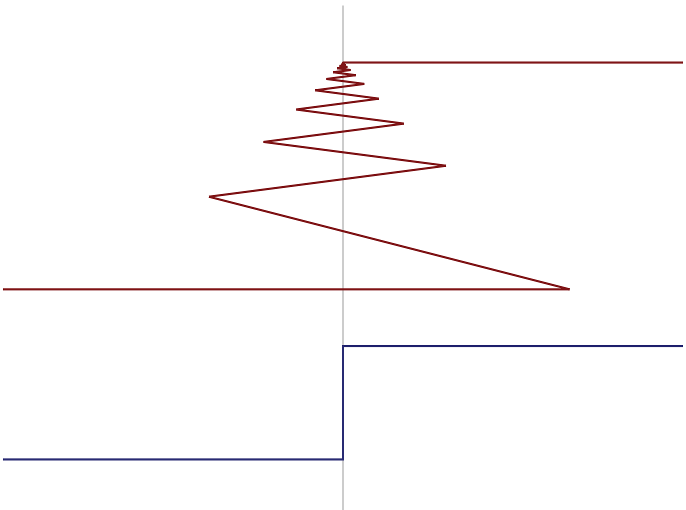

Fig. 5. A pair of paths such that the product of their end points is not connected by a path in their fiber product. Note that the upper path crosses the vertical line infinitely many times.

map from the curve to itself, we obtain a relation between points on the curve. To prove Theorem 1.1, we adapt the argument in the previous section to relations.

Recall that if $R \subset X \times Y$ and $R' \subset Y \times Z$ are the relations their composition $R' \circ R$ and the inverse R^{-1} are relations given by

$$R' \circ R = \{(x,z) : (x,y) \in R, (y,z) \in R' \text{ for some } y \in Y\},$$
$$R^{-1} = \{(y,x) : (x,y) \in R\}.$$

If $R \subset X \times X$ is a relation, a point $p \in X$ is a fixed point of R when $(p,p) \in R$. Let

$$\deg(f) = (\deg(f_1), \ldots, \deg(f_n)) \quad \text{for} \quad f = (f_1, \ldots, f_n) : \mathbb{S}^1 \to (\mathbb{S}^1 \times \cdots \times \mathbb{S}^1).$$

We will make use of the fiber product of graphs in a manner similar to [4]. Given maps $f_i : X_i \to Y$ for $i \in \{1, 2\}$, their fiber product is the set

$$X_1 \times_{f_1, f_2} X_2 = \{(x_1, x_2) \in X_1 \times X_2 : f_1(x_1) = f_2(x_2)\}.$$

When f_1 and f_2 are given by the same map f, respectively restricted to X_1 and X_2, we simply denote the fiber product by $X_1 \times_f X_2$.

If the domains of the maps $f_i : X_i \to Y$ are graphs $G_i = X_i$ (one-dimensional cell complexes), the target of the maps is either $Y = \mathbb{S}^1$ or $Y = \mathbb{R}^1$, and the f_i is injective on edges, then we define the fiber product to also be a graph given by the following cell decomposition: a point $(x_1, x_2) \in G_1 \times_{f_1, f_2} G_2$ is a vertex when either x_1 is a vertex of G_1 or x_2 is a vertex of G_2. The compliment of the vertices is then the disjoint union of interiors of edges. See Figure 4 for an example.

It is worth noting that the fiber product of two continuous surjections from the unit interval to itself with extrema fixed might not contain a path connecting the extrema. Such is the case in the example found in Figure 5.

Remark 4.1. Given $G = G_1 \times_{f_1, f_2} G_2$ satisfying the above: the G_i are graphs, the $f_i : G_i \to Y$ are injective on edges, and $Y \in \{\mathbb{S}^1, \mathbb{R}^1\}$; if $(x_1, x_2) = v \in G$ and $x_1 \in G_1$ are vertices but $x_2 \in G_2$ is not a vertex, then v and x_1 have the same graphical degree.

Proof. This follows from the observation that v and x_1 have homeomorphic neighborhoods contained in their respective stars. By the star of a vertex v, we mean the union of v and its adjacent edges. To see this, consider an open interval $N_2 \subset G_2$ around x_2, and let N_1 be the connected component of $f_1^{-1} \circ f_2(N_2)$ that contains x_1. We may choose

N_2 such that N_1 contains no vertices other than x_1. Let g_i be the restriction of f_i to N_i for $i \in \{1, 2\}$. Now the projection to the left factor gives a homeomorphism from the neighborhood $N_1 \times_{g_1,g_2} N_2$ of v to the neighborhood N_1 of x_1. ■

Lemma 4.2. Given relations $R_i \subset (\mathbb{S}^1 \times \mathbb{S}^1)$ and curves $\varphi_i : \mathbb{S}^1 \to R_i^\circ$ for $i \in \{1, 2\}$ with $\deg(\varphi_1) = (a, b_1)$ and $\deg(\varphi_2) = (b_2, c)$, if both b_1 and b_2 are odd, then there is a curve $\varphi : \mathbb{S}^1 \to (R_2 \circ R_1)^\circ$ with $\deg(\varphi) = (ka/b_1, kc/b_2)$ where k is an odd common multiple of b_1 and b_2.

For example, given

$$\varphi_1(\theta) = (\sin(\theta), \cos(\theta), \sin(3\theta), \cos(3\theta)),$$

$$\varphi_2(\theta) = (\sin(5\theta), \cos(5\theta), \sin(2\theta), \cos(2\theta)),$$

and R_i a neighborhood of φ_i, we have $\varphi \subset (R_2 \circ R_1)^\circ$ where

$$\varphi(\theta) = (\sin(5\theta), \cos(5\theta), \sin(6\theta), \cos(6\theta)).$$

Here $\deg(\varphi_1) = (1, 3)$, $\deg(\varphi_2) = (5, 2)$, and $\deg(\varphi) = (5, 6) = (k/3, 2k/5)$ with $k = 15$.

Proof. (Proof of Lemma 4.2) We denote the factors of the curves $\varphi_i : \mathbb{S}^1 \to \mathbb{T}^2$ by

$$\varphi_1 = (\varphi_{1x}, \varphi_{1y}) \text{ and } \varphi_2 = (\varphi_{2y}, \varphi_{2z}),$$

and we denote their respective domains by S_1 and S_2. We may assume that the curves φ_i are straight edge drawings of cycle graphs S_i. Otherwise replace φ_i with a straight edge approximation that has the same topological degrees and is sufficiently close to φ_i to be contained in R_i°. We may further assume that the y-coordinate of the vertices of φ_1 and φ_2 are all distinct. Let $G = S_1 \times_{\varphi_{1y},\varphi_{2y}} S_2$.

Since the values of φ_{1y} and φ_{2y} are distinct at every vertex of S_1 and S_2, we have for each vertex (s_1, s_2) of G that s_1 and s_2 cannot both be vertices. Hence by Remark 4.1, every vertex of G has graphical degree 2, so G must be a union of disjoint cycles. Choose $y_0 \in \mathbb{S}^1$ that is not the coordinate of any vertex of G, and consider the fiber $F \subset G$ above y_0,

$$F = \{(s_1, s_2) \in S_1 \times S_2 : \varphi_{1y}(s_1) = \varphi_{2y}(s_2) = y_0\}.$$

Since b_1 is odd, $|\varphi_{1\mathrm{y}}^{-1}(y_0)|$ is odd, and since b_2 is odd, $|\varphi_{2\mathrm{y}}^{-1}(y_0)|$ is odd, so $|F| = |\varphi_{1\mathrm{y}}^{-1}(y_0)||\varphi_{2\mathrm{y}}^{-1}(y_0)|$ is odd. Therefore, at least one of the cycles of G must intersect the fiber F in an odd number of points, and this cycle is the image of a simple closed curve $\sigma = (\sigma_1, \sigma_2) : \mathbb{S}^1 \to S_1 \times S_2$ that crosses F an odd number of times. There exists

$$\xi : \mathbb{S}^1 \to \mathbb{S}^1, \text{ such that } \xi(t) = \varphi_{1\mathrm{y}} \circ \sigma_1(t) = \varphi_{2\mathrm{y}} \circ \sigma_2(t),$$

where the 2nd equality holds since the range of σ is in G. Let k be the topological degree of ξ. The map ξ crosses y_0 an odd number of times, so k is odd.

Recall that the topological degree of a composition of maps is the product of their degrees. Since $\deg(\varphi_{1\mathrm{y}}) = b_1$ and $\deg(\varphi_{1\mathrm{y}}) \deg(\sigma_1) = \deg(\xi) = k$, we have $\deg(\sigma_1) = k/b_1$, and similarly $\deg(\sigma_2) = k/b_2$. Furthermore, since degree is integer valued, k is a common multiple of b_1 and b_2. Let

$$\varphi(t) = (\varphi_{1\,\mathrm{x}} \circ \sigma_1(t), \varphi_{2\mathrm{z}} \circ \sigma_2(t)).$$

Observe that $\varphi(t) \in (R_2 \circ R_1)^\circ$, since the range of σ is in G. Since $\deg(\varphi_{1\mathrm{x}}) = a$, and $\deg(\varphi_{2\mathrm{z}}) = c$, we have $\deg(\varphi) = (ka/b_1, kc/b_2)$. ■

Lemma 4.3. For a curve $\varphi : \mathbb{S}^1 \to \mathbb{T}^2$ with $\deg(\varphi) = (a, b)$, if $a \neq b$, then φ intersects the diagonal $D = \{(u, u) : u \in \mathbb{S}^1\} \subset \mathbb{T}^2$, and hence any relation R containing φ has a fixed point.

Proof. There is a deformation retraction ρ_t from $\mathbb{T}^2 \setminus D$ to the curve $\psi(u) = (u, -u)$ with $\deg(\psi) = (1, 1)$ given by

$$\rho_t(u, v) = \left(u, \tfrac{w_t}{\|w_t\|}\right), \text{ where } w_t = t(-u) + (1-t)v.$$

If a curve $\psi' : \mathbb{S}^1 \to \mathbb{T}^2$ avoids D, then ψ' can be deformed by ρ to be in ψ, so $\deg(\psi')$ is a multiple of $\deg(\psi)$, which means $\deg(\psi') = (a, a)$ for some $a \in \mathbb{Z}$. Thus, any curve φ with $\deg(\varphi) = (a, b)$ for $a \neq b$ cannot avoid the diagonal. ■

4.2 Proof of Theorem 1.1

Assume there is a simple closed curve $\gamma : \mathbb{S}^1 \to \mathbb{R}^d$ such that the i-th coordinate shadow $\pi_i(\gamma) = \lambda_i$ is a path $\lambda_i : \mathrm{I} \to \{x_i{=}0\} \subset \mathbb{R}^d$ for $i = 1, 2, 3$, and split γ into two closed arcs T_i and B_i (top and bottom) so that $\pi_i(T_i) = \pi_i(B_i) = \pi_i(\gamma)$ and $T_i \cap B_i = \{a_i, \tilde{a}_i\}$, where

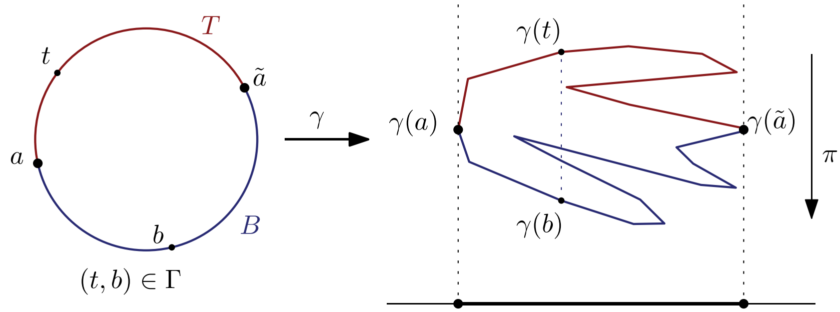

Fig. 6. The top and bottom arcs, and the relation Γ.

$\pi_i(a_i)$ and $\pi_i(\tilde{a}_i)$ are the end points of $\underline{\lambda}_i$. We choose labels so that a closed curve $\mathbb{S}^1 \to \underline{\gamma}$ that first traverses T_i from a_i to $\tilde{a}_i$ then traverses B_i from $\tilde{a}_i$ back to a_i has topological degree 1.

We define the relation $\Gamma_i \subset \underline{\gamma} \times \underline{\gamma}$ between a point in T_i and a point in B_i if they are mapped to the same point (see Figure 6). For $\varepsilon > 0$ let

$$\tilde{\Gamma}_{i,\varepsilon} = \left\{(t,b) \in T_i \times B_i : |\lambda_i^{-1}(\pi_i(t)) - \lambda_i^{-1}(\pi_i(b))| \leq \varepsilon\right\},$$

$$N_\varepsilon(p) = \{(x,y) \in \underline{\gamma} \times \underline{\gamma} : \|x - p\| + \|y - p\| \leq \varepsilon\},$$

$$\Gamma_{i,\varepsilon} = \tilde{\Gamma}_{i,\varepsilon} \cup \tilde{\Gamma}_{i,\varepsilon}^{-1} \cup N_\varepsilon(a_i) \cup N_\varepsilon(\tilde{a}_i).$$

Note that the $\Gamma_{i,\varepsilon}$ are a nested family of compact sets with $\cap_{\varepsilon>0}\Gamma_{i,\varepsilon} = \Gamma_i$, and that a_i and $\tilde{a}_i$ are the only two fixed points of Γ_i.

To show that $\Gamma_3 \circ \Gamma_2 \circ \Gamma_1$ has a fixed point, we prove a series of claims below.

Claim 4.4. For $\varepsilon > 0$ and for each $i \in \{1, 2, 3\}$, there is a curve $\psi_{i,\varepsilon} : \mathbb{S}^1 \to \Gamma_{i,\varepsilon}^\circ$ such that $\deg(\psi_{i,\varepsilon}) = (1, -1)$.

Proof. Since $\underline{\gamma}$ is contained in the curved surface $\pi_i^{-1}(\underline{\lambda}_i) = \underline{\lambda}_i + \mathbb{R}e_i$, we can factor $\underline{\gamma}$ through a map into the plane $\mathbb{R}^2$ by "unraveling" the surface. Specifically, let

$$\eta_i : \underline{\gamma} \to (\mathrm{I} \times \mathbb{R}), \quad \eta_i(x) = \left(\lambda_i^{-1} \circ \pi_i(x), e_i^*(x)\right).$$

Observe that for every $x, y \in \underline{\gamma}$,

$$\pi_i(x) = \pi_i(y) \quad \Longleftrightarrow \quad e_1^*(\eta_i(x)) = e_1^*(\eta_i(y)).$$

Let $\zeta_{i,\varepsilon} : \underline{\gamma} \to (\mathrm{I} \times \mathbb{R})$ such that $T'_{i,\varepsilon} = \zeta_{i,\varepsilon}(T_i)$ and $B'_{i,\varepsilon} = \zeta_{i,\varepsilon}(B_i)$ are polygonal paths with end points $a'_i = \eta_i(a_i)$ and $\tilde{a}'_i = \eta_i(\tilde{a}_i)$ that are sufficiently close to $\eta_i(T_i)$ and $\eta_i(B_i)$ that $G_{i,\varepsilon} = T'_{i,\varepsilon} \times_{e_1^*} B'_{i,\varepsilon} \subset \eta_i(\tilde{\Gamma}^\circ_{i,\varepsilon})$. We may obtain such a $\zeta_{i,\varepsilon}$, by choosing $T'_{i,\varepsilon}$ and $B'_{i,\varepsilon}$ to be polygonal $(\varepsilon/2)$-approximations of $\eta_i(T_i)$ and $\eta_i(B_i)$, so that for all $(t, b) \in \zeta^{-1}_{i,\varepsilon}(G_{i,\varepsilon})$ with $y = e_1^*(\zeta_{i,\varepsilon}(t)) = e_1^*(\zeta_{i,\varepsilon}(b))$, we have

$$\begin{aligned}\left|\lambda_i^{-1}(\pi_i(t)) - \lambda_i^{-1}(\pi_i(b))\right| &\le \left|y - \lambda_i^{-1}(\pi_i(t))\right| + \left|y - \lambda_i^{-1}(\pi_i(b))\right| \\ &\le \|\zeta_{i,\varepsilon}(t) - \eta_i(t)\| + \|\zeta_{i,\varepsilon}(b) - \eta_i(b)\| \\ &< \varepsilon.\end{aligned}$$

For simplicity, we choose $\zeta_{i,\varepsilon}$ so that the 1st coordinate of the vertices of $T'_{i,\varepsilon}$ and $B'_{i,\varepsilon}$ are all distinct, except at the end points a'_i, $\tilde{a}'_i$.

Since a'_i has a single adjacent edge in $T'_{i,\varepsilon}$ and a single adjacent edge in $B'_{i,\varepsilon}$, the vertex (a'_i, a'_i) of $G_{i,\varepsilon}$ has graphical degree 1. Likewise $(\tilde{a}'_i, \tilde{a}'_i)$ has graphical degree 1. By Remark 4.1, every other vertex of $G_{i,\varepsilon}$ has graphical degree 2. Hence, there is a path $\rho_{i,\varepsilon} : \mathrm{I} \to G_{i,\varepsilon}$ from $\rho_{i,\varepsilon}(0) = (a', a')$ to $\rho_{i,\varepsilon}(1) = (\tilde{a}', \tilde{a}')$.

Let $\sigma_{i,\varepsilon} : \mathrm{I} \to \tilde{\Gamma}^\circ_{i,\varepsilon}$ be $\sigma_{i,\varepsilon} = \eta_i^{-1} \circ \rho_{i,\varepsilon}$, and define $\sigma' : \mathrm{I} \to \tilde{\Gamma}^{-1\circ}_{i,\varepsilon}$ by $\sigma'(t) = (y, x)$ where $(x, y) = \sigma(1 - t)$, so that σ traverses $\tilde{\Gamma}^\circ_{i,\varepsilon}$ from (a_i, a_i) to $(\tilde{a}_i, \tilde{a}_i)$ and σ' traverses $\tilde{\Gamma}^{-1\circ}_{i,\varepsilon}$ from $(\tilde{a}_i, \tilde{a}_i)$ to (a_i, a_i). Now we define $\psi_{i,\varepsilon} : \mathbb{S}^1 \to \Gamma^\circ_{i,\varepsilon}$ to be the closed curve that first follows $\sigma_{i,\varepsilon}$ and then follows $\sigma'_{i,\varepsilon}$; that is,

$$\psi_{i,\varepsilon}(u) = \begin{cases} \sigma_{i,\varepsilon}(\theta_u/\pi) & \text{if } \theta_u \in [0, \pi] \\ \sigma'_{i,\varepsilon}((\theta_u/\pi) - 1) & \text{if } \theta_u \in [\pi, 2\pi] \end{cases}$$

where θ_u is the angle of the vector $u \in \mathbb{S}^1$.

Observe that the 1st factor of $\psi_{i,\varepsilon}$ traverses T_i from a_i to $\tilde{a}_i$ and then traverses B_i in the opposite direction, so the 1st factor has topological degree 1. Meanwhile, the 2nd factor of $\psi_{i,\varepsilon}$ traverses B_i from a_i to $\tilde{a}_i$ and then traverses T_i in the opposite direction, so the 2nd factor has topological degree − 1. Together this gives $\deg(\psi_{i,\varepsilon}) = (1, -1)$. ■

Claim 4.5. $\Gamma_{3,\varepsilon} \circ \Gamma_{2,\varepsilon} \circ \Gamma_{1,\varepsilon}$ has a fixed point.

Proof. Let $\psi_{i,\varepsilon}$ be the curves as in Claim 4.4 for $i = 1, 2, 3$. By Lemma 4.2 applied to $\psi_{1,\varepsilon}$ and $\psi_{2,\varepsilon}$ there is a curve $\psi' : \mathbb{S}^1 \to (\Gamma_{2,\varepsilon} \circ \Gamma_{1,\varepsilon})^\circ$ such that $\deg(\psi') = (j, j)$ for some odd integer j, and by Lemma 4.2 applied to ψ' and $\psi_{3,\varepsilon}$ there is a curve $\psi_\varepsilon : \mathbb{S}^1 \to (\Gamma_{3,\varepsilon} \circ \Gamma_{2,\varepsilon} \circ \Gamma_{1,\varepsilon})^\circ$ such that $\deg(\psi_\varepsilon) = (k, -k)$ for k an odd multiple of j. Therefore by Lemma 4.3, $\Gamma_{3,\varepsilon} \circ \Gamma_{2,\varepsilon} \circ \Gamma_{1,\varepsilon}$ has a fixed point. ■

Claim 4.6. $\Gamma_3 \circ \Gamma_2 \circ \Gamma_1$ has a fixed point.

Proof. Let $q_{0,k}$ be a fixed point of $\Gamma_{3,\varepsilon_k} \circ \Gamma_{2,\varepsilon_k} \circ \Gamma_{1,\varepsilon_k}$ for $\varepsilon_k \to 0$ monotonically and $(q_{0,k}, q_{1,k}) \in \Gamma_{1,\varepsilon_k}$ and $(q_{1,k}, q_{2,k}) \in \Gamma_{2,\varepsilon_k}$ and $(q_{2,k}, q_{0,k}) \in \Gamma_{3,\varepsilon_k}$. We may assume $q_{i,k} \to q_i$, since $\gamma \times \gamma$ is compact, otherwise restrict to a convergent subsequence for each $i \in \{0, 1, 2\}$. Since the $\Gamma_{1,\varepsilon}$ are nested, $(q_{0,j}, q_{1,j}) \in \Gamma_{1,\varepsilon_k}$ for $j \geq k$, and since Γ_{1,ε_k} is compact $(q_0, q_1) \in \Gamma_{1,\varepsilon_k}$. Hence $(q_0, q_1) \in \cap_{\varepsilon_k} \Gamma_{1,\varepsilon_k} = \Gamma_1$. Similarly $(q_1, q_2) \in \Gamma_2$ and $(q_2, q_0) \in \Gamma_3$. Thus, q_0 is a fixed point of $\Gamma_3 \circ \Gamma_2 \circ \Gamma_1$. ∎

Since $\Gamma_3 \circ \Gamma_2 \circ \Gamma_1$ has a fixed point, we get a contradiction in exactly the same way as we did in the last two paragraphs of the proof of Theorem 3.1 in Section 3. Namely, the vectors $q_1 - q_0$, $q_2 - q_1$, $q_0 - q_2$ are orthogonal to each other and sum to 0, so each vector must be 0. This completes the proof of Theorem 1.1.

5 Conclusion

5.1 Open questions

The above questions can be generalized to higher dimensions in other directions as well.

1. What is the maximum number of coordinate shadows of a k-sphere embedded in d-space that can be contractible?
2. What is the maximum number of coordinate shadows of a k-sphere embedded in d-space that can be embeddings of a k-ball?
3. What is the maximum number of coordinate shadows of a k-ball embedded in d-space that can be embeddings of a k-sphere?

For $d = k + 2$ the answer to question 1 is d by an inductive construction, and only special cases of questions 2 and 3 for $k = 1$ and $d = 3$ were known [1].

5.2 Difficulties of generalization

It is also worth noting a difficulty in extending to the case where $\mathbb{S}^1$ is replaced with a higher dimensional sphere. If $\gamma : \mathbb{S}^n \to \mathbb{R}^d$ is an embedding, then the preimage of the boundary of a shadow might not separate $\mathbb{S}^n$ into multiple component. Therefore, there does not seem to be a natural way to "flip" the sphere.

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