

BRANCHED COVERINGS OF SIMPLY CONNECTED MANIFOLDS

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ABSTRACT. We construct branched double coverings by certain direct products of manifolds for connected sums of copies of sphere bundles over the 2-sphere. As an application we answer a question of Kotschick and Löh up to dimension five. More precisely, we show that

- (1) every simply connected, closed four-manifold admits a branched double covering by a product of the circle with a connected sum of copies of $S^2 \times S^1$, followed by a collapsing map;
- (2) every simply connected, closed five-manifold admits a branched double covering by a product of the circle with a connected sum of copies of $S^3 \times S^1$, followed by a map whose degree is determined by the torsion of the second integral homology group of the target.

1. INTRODUCTION

The realization of manifolds as branched coverings is a classical long-standing problem in topology. A well-known theorem of Alexander [1] states that every oriented, closed, smooth n -dimensional manifold is a branched covering of S^n . Strong restrictions for the existence of branched coverings were found by Berstein and Edmonds [3]. Branched coverings have been investigated in many different contexts and they turned out to be a useful tool for the study of several problems in geometry, such as the minimal genus problem in four dimensions.

Recall that a branched d -fold covering is a smooth proper map $f: X \rightarrow Y$ with a codimension two subcomplex $B_f \subset Y$, called the branch locus of f , such that $f|_{X \setminus f^{-1}(B_f)}: X \setminus f^{-1}(B_f) \rightarrow Y \setminus B_f$ is a d -fold covering in the usual sense and for each $x \in f^{-1}(B_f)$ the map f is given by $(z, v) \mapsto (z^m, v)$, for some charts of x and $f(x)$ and some positive integer m . The point x is called singular and its image $f(x)$ is called a branch point.

In dimensions two and three, Edmonds showed that a dominant map is quite often homotopic to a branched covering [9, 10]. More precisely, Edmonds proved that every non-zero degree map between two closed surfaces is homotopic to the composition of a pinch map followed by a branched covering [9]. A pinch map in dimension two is a map which collapses 2-handles, i.e. is a quotient map $\pi: \Sigma \rightarrow \Sigma/\Sigma'$, where a submanifold $\Sigma' \subset \Sigma$, with circle boundary in the interior of Σ , is identified to a point. In dimension three, Edmonds result is that every π_1 -surjective map of degree at least three is homotopic to a branched covering [10]. The existence of branched coverings in low dimensions has been explored by several other people, including Fox, Hilden, Hirsch and Montesinos.

Our aim in this paper is to show that every simply connected, closed four-manifold, resp. five-manifold, admits a branched double covering by a product $S^1 \times N$, composed with a certain pinch map, resp. a map whose degree depends on the torsion of the integral homology of the target.

The main result is the following general statement:

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Theorem 1. *For $n \geq 4$ and every k there is a branched double covering*

$$S^1 \times (\#_k S^{n-2} \times S^1) \longrightarrow \#_k(S^{n-2} \widetilde{\times} S^2),$$

where $S^{n-2} \widetilde{\times} S^2$ denotes the total space of the non-trivial S^{n-2} -bundle over S^2 with structure group $\mathrm{SO}(n-1)$.

We note that the statement of Theorem 1 does not contain any reference on orientations, because the targets are simply connected and therefore always orientable.

The existence of dominant maps, not necessarily branched coverings, where the domain is a non-trivial product has an independent interest and its study is partially motivated by Gromov's work on functorial semi-norms on homology [13]. Kotschick and Löh [15] investigated such maps and showed that many targets with suitably large fundamental groups are not dominated by products. On the other hand, the fundamental group conditions given in [15] are not always sufficient to deduce domination by products, cf. [16, 17].

However, as pointed out in [15], it is natural to ask whether every connected, oriented, closed manifold with finite fundamental group is dominated by a non-trivial product. In order to study this question, it obviously suffices to obtain an answer for simply connected targets. In dimensions two and three, the answer is easy and affirmative, because S^2 and S^3 respectively represent the only simply connected manifolds in those two dimensions.

In this paper, we combine our constructions of branched coverings, with some classification results for simply connected manifolds in dimensions four and five (cf. [28, 11] and [23, 2] respectively) to obtain the following:

Theorem 2. *Every connected, oriented, closed manifold with finite fundamental group in dimensions four and five is dominated by a non-trivial product.*

In dimension four, Kotschick and Löh [15] have previously obtained a non-constructive proof for the above statement based on a result of Duan and Wang [8]. Our proof here is independent of those earlier works and it moreover gives an explicit construction of a degree two map from a product of type $S^1 \times (\#S^2 \times S^1)$ to every simply connected four-manifold, obtained as the composition of a branched double covering with a certain degree one map; cf. Theorem 4.

As we shall see in the course of the proof of Theorem 4, the (stable) diffeomorphism classification of Wall [28] and Freedman [11] will give us the desired degree one map $\mathbb{C}\mathbb{P}^2 \# \overline{\mathbb{C}\mathbb{P}^2} \longrightarrow M$. It is well-known that the homotopy classification of simply connected four-manifolds alone implies the existence of such a degree one map between the homotopy types of $\mathbb{C}\mathbb{P}^2 \# \overline{\mathbb{C}\mathbb{P}^2}$ and M . However, we appeal to the (stable) diffeomorphism classification theorems in order to obtain a considerably stronger result where all our dominant maps are smooth (whenever M itself is smooth). With those smooth maps in hand, we will be able to compare our construction with the aforementioned results of Edmonds in dimensions two and three; see the discussion following Remark 3 in Section 3 and Section 5.2.

Outline. In Section 2, we first prove Theorem 1 and then we discuss briefly the existence of branched double coverings for connected sums of direct products with a sphere factor. In Sections 3 and 4, we apply the constructions of Section 2 to simply connected four- and five-manifolds respectively in order to prove Theorem 2. In the final Section 5, we give another application in higher dimensions and we make further remarks.

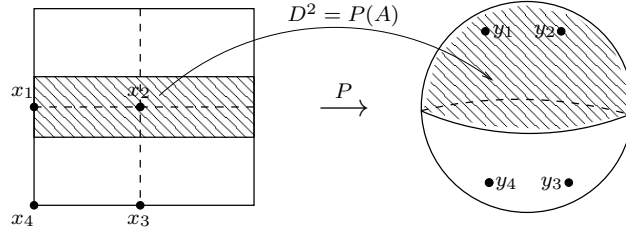


FIGURE 1. The 2-torus is a branched double covering of the 2-sphere, $P: T^2 \rightarrow S^2$, branched along four points $P(x_i) = y_i$, $i \in \{1, 2, 3, 4\}$, and so that $P(A) = D^2$, where A is an annulus in T^2 containing two singular points.

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2. CONSTRUCTION OF BRANCHED DOUBLE COVERINGS

We begin this section by proving Theorem 1. After that, we give a branched double covering for every connected sum of copies of a direct product $M \times S^2$, where M is any oriented, closed n -dimensional manifold (cf. Theorem 3).

Proof of Theorem 1. By assumption, we consider oriented S^{n-2} -bundles over S^2 with structure group $\mathrm{SO}(n-1)$, where $n \geq 4$. These bundles are classified by $\pi_1(\mathrm{SO}(n-1)) = \mathbb{Z}_2$ (cf. Steenrod [25]), therefore there exist only two; the product $S^{n-2} \times S^2$ and the twisted bundle $S^{n-2} \tilde{\times} S^2$. Let $\pi: S^{n-2} \tilde{\times} S^2 \rightarrow S^2$ denote the twisted bundle. There is a branched double covering with four branch points, $P: T^2 \rightarrow S^2$, given by the quotient for an involution on T^2 ; see Figure 1 (this map is known as “pillowcase”). We pull back π by P to obtain a branched double covering $P^*: P^*(S^{n-2} \tilde{\times} S^2) \rightarrow S^{n-2} \tilde{\times} S^2$. Now $P^*(S^{n-2} \tilde{\times} S^2)$ is the total space of an oriented S^{n-2} -bundle over T^2 with structure group $\mathrm{SO}(n-1)$. Again, there exist only two such bundles and since P has even degree we deduce that $P^*(S^{n-2} \tilde{\times} S^2)$ is the trivial bundle, i.e. the product $T^2 \times S^{n-2}$. Therefore, $T^2 \times S^{n-2}$ is a branched double covering of $S^2 \tilde{\times} S^{n-2}$, with branch locus four copies of the S^{n-2} -fiber of $S^2 \tilde{\times} S^{n-2}$, given by the preimages under π of the four branch points of P . This proves the statement of Theorem 1 for $k = 1$.

Next, we prove the claim for $k \geq 2$. Let the branched covering $P^*: T^2 \times S^{n-2} \rightarrow S^2 \tilde{\times} S^{n-2}$ constructed above. We can think of $T^2 \times S^{n-2}$ as a trivial S^1 -bundle over $S^1 \times S^{n-2}$. We thicken an S^1 -fiber of this bundle to an annulus A in T^2 so that $P(A) = D^2$, as in Figure 1. A fibered neighborhood of this S^1 -fiber is a product $S^1 \times D^{n-1}$ in $T^2 \times S^{n-2}$ and $P^*(S^1 \times D^{n-1})$ is an n -disk D^n in $S^2 \tilde{\times} S^{n-2}$. We now remove the fibered neighborhood $S^1 \times D^{n-1}$ from two copies of $T^2 \times S^{n-2}$ and perform a fiber sum by gluing the $S^1 \times S^{n-2}$ -boundaries. Since $T^2 \times S^{n-2}$ is a trivial S^1 -bundle this fiber sum will produce another trivial S^1 -bundle, namely a product $S^1 \times ((S^1 \times S^{n-2}) \# (S^1 \times S^{n-2}))$. At the same time, we connected sum two copies of $S^2 \tilde{\times} S^{n-2}$ along D^n , so that the branch loci fit together. We have now obtained a branched double covering

$$S^1 \times ((S^1 \times S^{n-2}) \# (S^1 \times S^{n-2})) \rightarrow (S^2 \tilde{\times} S^{n-2}) \# (S^2 \tilde{\times} S^{n-2}),$$

proving the claim for $k = 2$. For $k > 2$ we iterate the above construction. \square

An immediate consequence of Theorem 1 is the following:

Corollary 1. *For $n \geq 4$ and $k \geq 0$, every connected sum $\#_k(S^2 \widetilde{\times} S^{n-2})$ is a quotient space of the direct product $S^1 \times (\#_k S^{n-2} \times S^1)$.*

The statement of Theorem 1 is the most general possible concerning oriented S^{n-2} -bundles over S^2 , but it moreover presupposes the structure group of these bundles to be linear. This assumption is not necessary in dimensions four and five (cf. [22] and [14] respectively) and so we will be able to construct many branched coverings in those two dimensions; cf. Sections 3 and 4.

The inspiration for the construction of the branched covering of Theorem 1 stems from a previous joint work with Kotschick on domination for three-manifolds by circle bundles [16]. More precisely, we proved there that for every k there is a π_1 -surjective branched double covering $S^1 \times (\#_k S^1 \times S^1) \rightarrow \#_k(S^1 \times S^2)$. A direct generalization of that construction is achieved if we replace S^1 in $\#_k(S^1 \times S^2)$ by any oriented, closed n -dimensional manifold M . The steps of the proof follow those of [16] and they are left to the reader:

Theorem 3. *Let M be a connected, oriented, closed n -dimensional manifold. For every k there is a π_1 -surjective branched double covering $S^1 \times (\#_k M \times S^1) \rightarrow \#_k(M \times S^2)$.*

Remark 1. The constructions given in Theorems 1 and 3 are of different nature, because the connected summands of the target in Theorem 1 are twisted products, while in Theorem 3 the summands are direct products. For this reason the proof of Theorem 1 cannot be deduced using Theorem 3. (Nevertheless, Theorem 3 could be seen as a special case of Theorem 1 for $M = S^n$.) A further generalization of Theorem 1 would be to consider targets that are connected sums of fiber bundles over S^2 , where the fiber is an arbitrary oriented, closed manifold M . However, the comprehension of arbitrary twisted products $S^2 \widetilde{\times} M$ (and of connected sums built out of such summands) seems, in general, not sufficiently enough to produce further generalization of Theorem 1.

Remark 2. The two-dimensional pillowcase $T^2 = S^1 \times S^1 \rightarrow S^2$ can be generalized for any sphere S^n , $n \geq 2$. Namely, for every $n > k \geq 1$, there is a branched double covering $P: S^k \times S^{n-k} \rightarrow S^n$. The branched locus of P is $B_P = S^{k-1} \times S^{n-k-1}$ and there is an n -ball $D^n \subset S^n$ such that $P^{-1}(D^n) = S^k \times D^{n-k}$. Therefore, in place of S^2 in Theorem 3 we may take any sphere of dimension at least two. We refer to Chapter 3 of [17] for further details.

3. SIMPLY CONNECTED FOUR-MANIFOLDS

We now apply Theorem 1 to simply connected four-manifolds. First, we show that every four-manifold diffeomorphic to a connected sum $\#_k \mathbb{C}\mathbb{P}^2 \#_k \overline{\mathbb{C}\mathbb{P}^2}$ admits a branched double covering by a product $S^1 \times (\#_k S^1 \times S^2)$. We then deduce by the classification results of Wall [28] and Freedman [11], that every simply connected, closed four-manifold admits a branched double covering by a product of the circle with a connected sum of copies of $S^2 \times S^1$, followed by a collapsing map.

By a result of Smale [22], the inclusion $\mathrm{SO}(3) \hookrightarrow \mathrm{Diff}^+(S^2)$ is a homotopy equivalence and so there exist only two oriented S^2 -bundles over S^2 . Moreover, $\mathbb{C}\mathbb{P}^2 \#_k \overline{\mathbb{C}\mathbb{P}^2}$ is diffeomorphic to $S^2 \widetilde{\times} S^2$ (cf. Wall [27]), therefore Theorem 1 implies the following for $n = 4$:

Corollary 2. *For every k there is a branched double covering $S^1 \times (\#_k S^1 \times S^2) \rightarrow \#_k \mathbb{C}\mathbb{P}^2 \#_k \overline{\mathbb{C}\mathbb{P}^2}$.*

Given a connected sum $P \# Q$, we define a pinch map $P \# Q \rightarrow P$, by collapsing the gluing sphere and then mapping Q to a point. This degree one map is called collapsing map. For a

connected sum $P\#\overline{\mathbb{C}\mathbb{P}^2}$, instead of the whole summand $\overline{\mathbb{C}\mathbb{P}^2}$, we may collapse the exceptional embedded sphere $\overline{\mathbb{C}\mathbb{P}^1} \subset \overline{\mathbb{C}\mathbb{P}^2}$ to obtain again a degree one map $P\#\overline{\mathbb{C}\mathbb{P}^2} \rightarrow P$. The collapsed $\overline{\mathbb{C}\mathbb{P}^1}$ has self-intersection number -1 and is called -1 -sphere. If P is smooth, then this is the usual blow-down operation. Similarly, for a connected sum $P\#\mathbb{C}\mathbb{P}^2$ we obtain a degree one map $P\#\mathbb{C}\mathbb{P}^2 \rightarrow P$ by collapsing the embedded $+1$ -sphere $\mathbb{C}\mathbb{P}^1 \subset \mathbb{C}\mathbb{P}^2$. In the smooth category, this operation is known as the antiblow-down of P .

In the light of Corollary 2, we now rely on results of Wall [28] and Freedman [11] on the classification of simply connected, closed four-manifolds to obtain the following statement which moreover proves Theorem 2 in dimension four:

Theorem 4. *Every simply connected, closed four-manifold M admits a degree two map by a product $S^1 \times (\#_k S^2 \times S^1)$, which is given by the composition of a branched double covering $S^1 \times (\#_k S^2 \times S^1) \rightarrow \#_k \mathbb{C}\mathbb{P}^2 \#_k \overline{\mathbb{C}\mathbb{P}^2}$ with a collapsing map $\#_k \mathbb{C}\mathbb{P}^2 \#_k \overline{\mathbb{C}\mathbb{P}^2} \rightarrow M$.*

Proof. By Corollary 2, it suffices to show that for every simply connected, closed four-manifold M there exists a k and a collapsing map $\#_k \mathbb{C}\mathbb{P}^2 \#_k \overline{\mathbb{C}\mathbb{P}^2} \rightarrow M$.

First, if necessary, we perform connected sums of M with copies of $\mathbb{C}\mathbb{P}^2$ or $\overline{\mathbb{C}\mathbb{P}^2}$ (or both) to obtain a manifold $M\#_p \mathbb{C}\mathbb{P}^2 \#_q \overline{\mathbb{C}\mathbb{P}^2}$, whose intersection form is odd and indefinite (and therefore diagonal; cf. [18]).

If M is smooth, then Wall's stable diffeomorphism classification in dimension four [28] implies that $M\#_p \mathbb{C}\mathbb{P}^2 \#_q \overline{\mathbb{C}\mathbb{P}^2}$ is stably diffeomorphic to a connected sum $\#_l \mathbb{C}\mathbb{P}^2 \#_m \overline{\mathbb{C}\mathbb{P}^2}$. The connected summing with $\overline{\mathbb{C}\mathbb{P}^2}$, resp. $\mathbb{C}\mathbb{P}^2$, is the blow-up, resp. antiblow-up, operation.

For M non-smooth, we may first assume that the homotopy type of M has trivial Kirby-Siebenmann invariant and then deduce that $M\#_p \mathbb{C}\mathbb{P}^2 \#_q \overline{\mathbb{C}\mathbb{P}^2}$ is homeomorphic to a connected sum $\#_l \mathbb{C}\mathbb{P}^2 \#_m \overline{\mathbb{C}\mathbb{P}^2}$, by Freedman's topological classification of simply connected four-manifolds [11]. In particular, $M\#_p \mathbb{C}\mathbb{P}^2 \#_q \overline{\mathbb{C}\mathbb{P}^2}$ inherits a smooth structure.

We can now assume that, in both cases, $M\#_p \mathbb{C}\mathbb{P}^2 \#_q \overline{\mathbb{C}\mathbb{P}^2}$ is (stably) diffeomorphic to a connected sum $\#_k \mathbb{C}\mathbb{P}^2 \#_k \overline{\mathbb{C}\mathbb{P}^2}$, after connected summing with more copies of $\mathbb{C}\mathbb{P}^2$ or $\overline{\mathbb{C}\mathbb{P}^2}$, if necessary.

Finally, a degree one collapsing map

$$\#_k \mathbb{C}\mathbb{P}^2 \#_k \overline{\mathbb{C}\mathbb{P}^2} \cong M\#_p \mathbb{C}\mathbb{P}^2 \#_q \overline{\mathbb{C}\mathbb{P}^2} \rightarrow M$$

is obtained by collapsing the q embedded exceptional spheres $\overline{\mathbb{C}\mathbb{P}^1} \subset \overline{\mathbb{C}\mathbb{P}^2}$ and the p embedded spheres $\mathbb{C}\mathbb{P}^1 \subset \mathbb{C}\mathbb{P}^2$. If M is smooth, then the collapsing map is also smooth.

We have now completed the proof of Theorem 4 and therefore of Theorem 2 in dimension four. \square

Remark 3. Wall's stable diffeomorphism classification [28] is obtained after adding summands $S^2 \times S^2$. However, for simply connected, closed, smooth four-manifolds with odd intersection form (as in the proof of Theorem 4), this is equivalent to adding summands $\mathbb{C}\mathbb{P}^2 \# \overline{\mathbb{C}\mathbb{P}^2}$, again by a result of Wall [27]; see also Remark 4.

In dimension two, Edmonds [9] proved that every non-zero degree map between two closed surfaces is homotopic to the composition of a pinch map followed by a branched covering. We observe that the maps constructed in Theorem 4 for every simply connected, closed four-manifold

have an analogy to Edmonds result, where however the order between the pinch map and the branched covering is reversed.

Moreover, Edmonds [9] showed that a non-zero degree map between two closed surfaces, $f: \Sigma \rightarrow F$, is homotopic to a branched covering if and only if either f is π_1 -injective or $|\deg(f)| > [\pi_1(F): \pi_1(f)(\pi_1(\Sigma))]$. This implies that every non-zero degree map between two closed surfaces can be lifted to a (π_1 -surjective) map which is homotopic either to a pinch map (absolute degree one) or to a branched covering.

In dimension three, every π_1 -surjective map of degree greater than two between two closed three-manifolds is homotopic to a branched covering, again by a result of Edmonds [10]. This result fails in dimension four, by a recent work of Pankka and Souto [19], where it is shown that T^4 is not a branched covering of $\#_3(S^2 \times S^2)$, while every integer can be realized as the degree for a map from T^4 to $\#_3(S^2 \times S^2)$ (by a criterion of Duan and Wang [8]; see Section 5.2).

4. SIMPLY CONNECTED FIVE-MANIFOLDS

In this section we show that every closed five-manifold with finite fundamental group is dominated by products. We first recall the classification of simply connected, closed five-manifolds by Smale [23] and Barden [2] and show that every five-manifold diffeomorphic to a connected sum of copies of the two S^3 -bundles over S^2 admits a branched double covering by a product $S^1 \times (\#_k S^1 \times S^3)$. We then give some existence results of dominant maps between simply connected five-manifolds and using these results we prove Theorem 2 for five-manifolds.

Given two n -dimensional manifolds M and N with boundaries ∂M and ∂N respectively, we form a new manifold $M \cup_f N$, where f is an orientation reversing diffeomorphism of any $(n-1)$ -dimensional submanifold of ∂N with one of ∂M .

Smale [23] classified simply connected, spin five-manifolds and a few years later Barden [2] completed that classification by including non-spin manifolds as well. The following constructions are given in [2]: Let $S^3 \times S^2$, $S^3 \tilde{\times} S^2$ be the two S^3 -bundles over S^2 and $A = S^2 \times D^3$, $B = S^2 \tilde{\times} D^3$ be the two D^3 -bundles over S^2 with boundaries $\partial A = S^2 \times S^2$ and $\partial B = S^2 \tilde{\times} S^2 \cong \mathbb{C}\mathbb{P}^2 \# \overline{\mathbb{C}\mathbb{P}^2}$ respectively. As in dimension four, we don't need to assume that the structure group of oriented S^3 -bundles over S^2 is linear, because the inclusion $\mathrm{SO}(4) \hookrightarrow \mathrm{Diff}^+(S^3)$ is a homotopy equivalence, by the proof of the Smale conjecture (cf. Hatcher [14]).

A prime, simply connected, closed, spin five-manifold is either $M_1 := S^5$ or $M_\infty := S^2 \times S^3$, if its integral homology groups have no torsion. If its second homology group is torsion, then is

$$M_k := (A \#_{\partial} A) \cup_{\overline{f_k}} (\overline{A \#_{\partial} A}), 1 < k < \infty,$$

where $A \#_{\partial} A$ denotes the boundary connected sum of two copies of A and f_k is an orientation preserving diffeomorphism on $\partial(A \#_{\partial} A) = (S^2 \times S^2) \# (S^2 \times S^2)$, realizing a certain isomorphism

$$H_2(f_k; \mathbb{Z}): H_2(\partial(A \#_{\partial} A); \mathbb{Z}) \xrightarrow{\cong} H_2(\partial(A \#_{\partial} A); \mathbb{Z}).$$

The second integral homology of M_k is given by $H_2(M_k; \mathbb{Z}) = \mathbb{Z}_k \oplus \mathbb{Z}_k$, $1 < k < \infty$; cf. [2]. For details on the f_k 's construction see [27].

A prime, simply connected, closed, non-spin five-manifold with torsion-free integral homology is the non-trivial S^3 -bundle over S^2 , denoted by X_∞ . A simply connected, closed, non-spin five-manifold with torsion second integral homology is either

- $X_{-1} := B \cup_{g_{-1}} \overline{B}$, where g_{-1} is an orientation preserving diffeomorphism on ∂B , realizing a certain isomorphism

$$H_2(g_{-1}; \mathbb{Z}) : H_2(\partial B; \mathbb{Z}) \xrightarrow{\cong} H_2(\partial B; \mathbb{Z}),$$

or

- $X_m := (B \#_{\partial} B) \cup_{g_m} \overline{(B \#_{\partial} B)}$, $1 \leq m < \infty$, where $B \#_{\partial} B$ denotes the boundary connected sum of two copies of B and g_m is an orientation preserving diffeomorphism on $\partial(B \#_{\partial} B) = (S^2 \tilde{\times} S^2) \# (S^2 \tilde{\times} S^2)$, realizing a certain isomorphism

$$H_2(g_m; \mathbb{Z}) : H_2(\partial(B \#_{\partial} B); \mathbb{Z}) \xrightarrow{\cong} H_2(\partial(B \#_{\partial} B); \mathbb{Z}).$$

Except $X_1 \cong X_{-1} \# X_{-1}$, each X_m is prime. Their second integral homology group is given by $H_2(X_m; \mathbb{Z}) = \mathbb{Z}_{2^m} \oplus \mathbb{Z}_{2^m}$, $0 < m < \infty$ and $H_2(X_{-1}; \mathbb{Z}) = \mathbb{Z}_2$; cf. [2]. Also, X_{-1} is the Wu manifold $SU(3)/SO(3)$; cf. [6, 30]. Finally, we set $X_0 := S^5$ which is spin and has torsion-free homology.

According to the above data, we have the following classification theorem for simply connected five-manifolds.

Theorem 5 (Barden [2]). *Every simply connected, closed five-manifold M is diffeomorphic to a connected sum $M_{k_1} \# \dots \# M_{k_l} \# X_m$, where $-1 \leq m \leq \infty$, $l \geq 0$, $k_1 > 1$ and k_i divides k_{i+1} or $k_{i+1} = \infty$.*

Since every summand, except $X_1 \cong X_{-1} \# X_{-1}$, is prime, we may refer to the above decomposition of M as prime decomposition. Moreover, a summand $X_{m \neq 0}$ exists if and only if M is not spin.

For a simply connected, closed five-manifold with torsion-free homology the following particular classification result holds:

Theorem 6 (Smale [23], Barden [2]). *Every simply connected, closed five-manifold M with torsion-free second homology group $H_2(M; \mathbb{Z}) = \mathbb{Z}^k$ is (up to diffeomorphism)*

- (1) *either a connected sum $\#_k(S^3 \times S^2)$, for M spin, or*
- (2) *a connected sum $\#_{k-1}(S^3 \times S^2) \# (S^3 \tilde{\times} S^2)$, for M non-spin.*

Remark 4. By a theorem of Wall [27], a connected sum $\#_{k-1}(S^3 \times S^2) \# (S^3 \tilde{\times} S^2)$ is diffeomorphic to $\#_k(S^3 \tilde{\times} S^2)$. In Remark 3, we refer to the corresponding statement in dimension four. This is also true for all oriented S^{n-2} -bundles over S^2 , $n \geq 6$, with structure group $SO(n-1)$.

The above classification results will be our guide to prove Theorem 2 for five-manifolds. First, we can prove this theorem for every simply connected five-manifold with torsion-free second homology group:

Proposition 1. *Let M be a simply connected, closed five-manifold with torsion-free second homology group $H_2(M; \mathbb{Z}) = \mathbb{Z}^k$. Then M admits a branched double covering by the product $S^1 \times (\#_k S^1 \times S^3)$.*

Proof. We know by Theorem 6 and by Remark 4 that such M is diffeomorphic to a connected sum of copies of the twisted product $S^2 \tilde{\times} S^3$ or of the trivial bundle $S^3 \times S^2$. Now Theorem 1 and Theorem 3 imply the proof, for $\#_k(S^2 \tilde{\times} S^3)$ and $\#_k(S^3 \times S^2)$ respectively. \square

In the remainder of this section we complete the proof of Theorem 2 for five-manifolds. We observe that $S^2 \times S^3$ is a branched double covering of the other two prime, simply connected, closed five-manifolds with torsion-free homology, namely of $S^2 \widetilde{\times} S^3$ and S^5 . Indeed:

- A branched double covering $S^2 \times S^3 \longrightarrow S^2 \widetilde{\times} S^3$ is obtained by pulling back the S^3 -bundle map $S^2 \widetilde{\times} S^3 \longrightarrow S^2$ by a branched double covering $S^2 \longrightarrow S^2$ (recall that S^3 -bundles over S^2 are classified by $\pi_1(\mathrm{SO}(4)) = \mathbb{Z}_2$).
- $S^2 \times S^3$ is a branched double covering of S^5 , by pulling back the S^1 -bundle map $S^5 \longrightarrow \mathbb{C}\mathbb{P}^2$ by a branched double covering $S^2 \times S^2 \longrightarrow \mathbb{C}\mathbb{P}^2$; cf. [12, 7]. (A branched double covering $S^2 \times S^2 \longrightarrow \mathbb{C}\mathbb{P}^2$ is obtained as the quotient map for the involution $(x, y) \mapsto (y, x)$ of $S^2 \times S^2$.)

As we shall see below, every simply connected, closed five-manifold M with torsion second integral homology group admits a non-zero degree map by $S^2 \times S^3$, given by the composition of a branched double covering $S^2 \times S^3 \longrightarrow S^5$ with a dominant map $S^5 \longrightarrow M$ whose degree depends on $H_2(M; \mathbb{Z})$. The latter map will be obtained by applying the Hurewicz theorem modulo a Serre class of groups.

A Serre class of abelian groups is a non-empty class \mathcal{C} of abelian groups such that for every exact sequence $A \longrightarrow B \longrightarrow C$, where $A, C \in \mathcal{C}$, then $B \in \mathcal{C}$. A Serre class \mathcal{C} is called a ring of abelian groups if it is closed under the tensor and the torsion product operations. Moreover, \mathcal{C} is said to be acyclic if for every aspherical space X with $\pi_1(X) \in \mathcal{C}$, then the homology groups $H_i(X; \mathbb{Z}) \in \mathcal{C}$, for all $i > 0$. We say that two abelian groups A, B are isomorphic modulo \mathcal{C} if there is a homomorphism between A and B whose kernel and cokernel belong to \mathcal{C} .

The Hurewicz theorem modulo a Serre class states the following:

Theorem 7 (Serre [21]). *Let X be a simply connected space and \mathcal{C} be an acyclic ring of abelian groups. Then the following are equivalent:*

- $\pi_i(X) \in \mathcal{C}$, for all $1 < i < n$,
- $H_i(X) \in \mathcal{C}$, for all $1 < i < n$.

Moreover, each of the above statements implies that the Hurewicz homomorphism $h: \pi_i(X) \longrightarrow H_i(X)$ is an isomorphism modulo \mathcal{C} for all $i \leq n$.

As a consequence of this version of the Hurewicz theorem, every simply connected, closed n -dimensional manifold M whose homology groups $H_i(M; \mathbb{Z})$ are all k -torsion, for $0 < i < n$, is minimal with respect to the domination relation (i.e. it is dominated by every other manifold).

Corollary 3 (Ruberman [20]). *Let M be a simply connected, closed n -dimensional manifold with k -torsion homology groups $H_i(M; \mathbb{Z})$, for $0 < i < n$ and some integer k . Then the image of the Hurewicz homomorphism $\pi_n(M) \longrightarrow H_n(M)$ is given by $k^r \mathbb{Z}$ for some r . In particular, there is a map $S^n \longrightarrow M$ of degree k^r .*

We have now shown that every five-manifold M which is a connected sum of copies of M_k and X_m , where $1 \leq k < \infty$ and $-1 \leq m < \infty$, admits a branched double covering by the product $S^2 \times S^3$, followed by a map $S^5 \longrightarrow M$ whose degree is determined by $H_2(M; \mathbb{Z})$, being a power of the least common multiple of the torsion second integral homology groups $H_2(M_k; \mathbb{Z})$ and $H_2(X_m; \mathbb{Z})$. As we have seen above, the non-trivial S^3 -bundle over S^2 admits a branched double covering by the product $S^2 \times S^3$. We now want to combine these maps, together with our constructions from Section 2, to obtain domination by products for every simply connected, closed five-manifold.

We remark that it is not generally possible to obtain a non-zero degree map $M_1 \# M_2 \rightarrow N_1 \# N_2$ by connected summing any two non-zero degree maps between closed n -dimensional manifolds, $f_i: M_i \rightarrow N_i$, $i = 1, 2$. A first obstruction is that the preimage $f_i^{-1}(D_i^n)$ of a removed n -disk D_i^n from N_i , is not generally an n -disk in M_i . Moreover, even if we can find such a disk, we need the degrees of f_1 and f_2 to be equal in order to connected sum the domains and the targets, otherwise we cannot paste those maps along the gluing sphere (recall that maps between spheres are classified by their degrees).

If the above two constraints are satisfied, then we can paste those f_i together to obtain a new map $M_1 \# M_2 \rightarrow N_1 \# N_2$ of the same non-zero degree. The π_1 -surjectivity of the f_i is a sufficient condition to overcome the first obstacle (cf. [24]).

Lemma 1 (Derbez-Sun-Wang [5]). *Let M_i, N_i be connected, oriented, closed n -dimensional manifolds, $n \geq 3$, and assume that for $i = 1, \dots, k$ there exist π_1 -surjective maps $M_i \rightarrow N_i$ of non-zero degree d . Then there is a π_1 -surjective map $\#_{i=1}^k M_i \rightarrow \#_{i=1}^k N_i$ of degree d .*

For simply connected targets the π_1 -surjectivity condition is automatically satisfied. Moreover, $S^2 \times S^3$ and S^5 admit self-maps of any degree, which implies that the domination for every minimal summand by S^5 and for every S^3 -bundle over S^2 by $S^2 \times S^3$ can be done by maps of the same degree. We therefore obtain the following statement which also completes the proof of Theorem 2 for five-manifolds:

Theorem 8. *Let M be a simply connected, closed five-manifold so that $H_2(M; \mathbb{Z})$ has rank k . Then M admits a non-zero degree map by the product $S^1 \times (\#_k S^1 \times S^3)$, which is given by the composition of a branched double covering $S^1 \times (\#_k S^1 \times S^3) \rightarrow \#_k(S^2 \times S^3)$ with a map $\#_k(S^2 \times S^3) \rightarrow M$ whose degree is determined by the torsion of $H_2(M; \mathbb{Z})$.*

Proof. By Theorem 5, a simply connected, closed five-manifold M is diffeomorphic to a connected sum $M_{k_1} \# \dots \# M_{k_l} \# X_m$, where the summands M_{k_i}, X_m are described at the beginning of this section. Clearly, the rank k of $H_2(M; \mathbb{Z})$ is equal to the number of M_∞ and X_∞ , i.e. the number of S^3 -bundles over S^2 . Furthermore, we may assume that the torsion of $H_2(M; \mathbb{Z})$ is not trivial, otherwise we appeal to Proposition 1 to deduce that M admits a branched double covering by a product $S^1 \times (\#_k S^1 \times S^3)$.

By Corollary 3 (and the comments following that) we deduce that the part of $M_{k_1} \# \dots \# M_{k_l} \# X_m$ not containing M_∞ or X_∞ admits a dominant map by S^5 whose degree is a power of the least common multiple of the torsion second integral homology groups $H_2(M_{k_i}; \mathbb{Z})$ and $H_2(X_m; \mathbb{Z})$, $k_i, m \neq \infty$. Now Lemma 1 implies that M is dominated by a manifold diffeomorphic to $\#_k(S^2 \times S^3)$ (if $m = \infty$, we additionally use the fact that M_∞ is a branched double covering of X_∞ , as we have seen above in this section). The degree of $\#_k(S^2 \times S^3) \rightarrow M$ is clearly determined (up to multiplication by two) by the torsion of $H_2(M; \mathbb{Z})$. Finally, Theorem 3 implies that $\#_k(S^2 \times S^3)$ admits a branched double covering by $S^1 \times (\#_k S^1 \times S^3)$, completing the proof of Theorem 8. \square

Remark 5. By Remark 2, we also have that $\#_k(S^2 \times S^3)$ admits branched double coverings by $S^1 \times (\#_k S^2 \times S^2)$ and $S^2 \times (\#_k S^2 \times S^1)$, and branched four-fold coverings by $T^2 \times (\#_k S^1 \times S^2)$ and $S^1 \times S^2 \times \Sigma_k$, where Σ_k is an oriented, closed surface of genus k .

Remark 6. In contrast to Theorem 4, the statement of Theorem 8 does not provide absolute control of the degree of the map $\#_k(S^2 \times S^3) \rightarrow M$ following the branched double covering $S^1 \times (\#_k S^1 \times S^3) \rightarrow \#_k(S^2 \times S^3)$. This is related to the problem of determining the sets of (self-)mapping degrees of simply connected five-manifolds, which is essentially still unsolved; cf. [26]. However,

an aside application of Theorem 7 is that the sets of degrees of maps between the summands M_k and X_m (for $k, m \neq \infty$) are infinite, because every multiple of a power of the torsion second homology of X_m (resp. M_k) can be realized as a degree for a map $M_k \rightarrow X_m$ (resp. $X_m \rightarrow M_k$); recall that S^5 admits self-maps of any degree and it is minimal for the domination relation. In particular, we deduce, without using formality (see [4]), that the set of self-mapping degrees for every simply connected five-manifold is infinite.

Remark 7. Following the ideas of [26], one can show the existence of a non-zero degree map $\#_k(S^2 \times S^3) \rightarrow M$ using the cell decompositions of M and $\#_k(S^2 \times S^3)$. Namely, if M is a simply connected five-manifold with $\text{rank } H_2(M; \mathbb{Z}) = k$, then by [2] there is a simply connected three-dimensional cell complex X^3 with torsion second integral homology group such that

$$M = (S_1^2 \vee S_1^3 \vee \cdots \vee S_k^2 \vee S_k^3 \vee X^3) \cup_\varphi D^5.$$

The attaching map φ is given as the sum of the Whitehead products $[\iota_i, \eta_i]$ (where ι_i and η_i denote inclusions from S^2 and S^3 respectively) with a summand of finite order d which depends on the torsion of $H_2(M; \mathbb{Z})$. Moreover, the connected sum $\#_k(S^2 \times S^3)$ has a cell decomposition

$$\#_k(S^2 \times S^3) = (S_1^2 \vee S_1^3 \vee \cdots \vee S_k^2 \vee S_k^3) \cup_\psi D^5,$$

where ψ is the sum of the Whitehead products $[\iota_i, \eta_i]$. Let now the map

$$f: S_1^2 \vee S_1^3 \vee \cdots \vee S_k^2 \vee S_k^3 \rightarrow S_1^2 \vee S_1^3 \vee \cdots \vee S_k^2 \vee S_k^3 \vee X^3$$

given by $f|_{S_i^2} = d \cdot \iota_i$ and $f|_{S_i^3} = \eta_i$. Then for all $i = 1, \dots, k$ we obtain

$$f_*[\iota_i, \eta_i] = [f_*\iota_i, f_*\eta_i] = [d \cdot \iota_i, \eta_i] = d \cdot [\iota_i, \eta_i]$$

and so $f_*(\psi) = d \cdot \varphi$, implying that f extends to a map $\#_k(S^2 \times S^3) \rightarrow M$ of degree d ; see [26] for further details.

5. A FURTHER APPLICATION AND FINAL REMARKS

5.1. Six-manifolds. In the light of Theorems 1 and 3 we can further verify that the fundamental classes of certain simply connected manifolds in dimensions higher than five are representable by products.

As an example we deal with 2-connected, closed six-manifolds. Wall [29] classified simply connected, closed, smooth six-manifolds (see also [31]). As usual, the empty connected sum is S^6 .

Theorem 9 (Wall [29]). *Let M be a simply connected, closed, smooth six-manifold. Then M is diffeomorphic to $N \# (S^3 \times S^3) \# \cdots \# (S^3 \times S^3)$, where $H_3(N)$ is finite.*

If now the target is 2-connected then $N = S^6$; cf. Smale [23]. In that case the topological and the diffeomorphism classification coincide (cf. Wall [29]) and so we obtain:

Corollary 4. *Every 2-connected, closed six-manifold M admits a branched double covering by $S^1 \times (\#_k S^2 \times S^3)$, where $k = \frac{1}{2} \text{rank } H_3(M; \mathbb{Z})$.*

Proof. By the above results of Wall [29] and Smale [23], every 2-connected, closed six-manifold M is diffeomorphic to $\#_k(S^3 \times S^3)$, for some non-negative integer $k = \frac{1}{2} \text{rank } H_3(M; \mathbb{Z})$. The proof now follows by the generalized pillowcase map in Remark 2. \square

Remark 8. By Remark 2, M admits also a branched double covering by $S^2 \times (\#_k S^1 \times S^3)$, and branched four-fold coverings by $T^2 \times (\#_k S^2 \times S^2)$, $S^1 \times S^2 \times (\#_k S^2 \times S^1)$ and $S^2 \times S^2 \times \Sigma_k$.

We note that M admits a degree two map by $S^1 \times (\#_k T^5)$ as well. For take the branched double covering $S^1 \times (\#_k S^2 \times S^3) \rightarrow M$ of Corollary 4 and precompose it with a degree one map $S^1 \times (\#_k T^2 \times T^3) \rightarrow S^1 \times (\#_k S^2 \times S^3)$, using Lemma 1 and the fact that every sphere is minimal admitting maps of any degree.

5.2. Branched coverings in dimension four. Every dominant map onto a simply connected four-manifold

$$(1) \quad S^1 \times (\#_k S^2 \times S^1) \rightarrow N,$$

is a priori π_1 -surjective because the target has trivial fundamental group. However, the branched double covering $S^1 \times (\#_k S^2 \times S^1) \rightarrow \#_k(S^2 \times S^2)$ obtained in Theorem 3 (for $M = S^2$) is π_1 -surjective also by construction. In a previous joint work with Kotschick [16] we have shown that there is a branched double covering

$$(2) \quad S^1 \times \Sigma_k \rightarrow \#_k(S^2 \times S^1)$$

which is again π_1 -surjective (now the target $\#_k(S^2 \times S^1)$ is not simply connected). We multiply (2) with the identity map on S^1 to obtain a π_1 -surjective branched double covering

$$(3) \quad T^2 \times \Sigma_k \rightarrow S^1 \times (\#_k S^2 \times S^1).$$

Now we compose (3) with (1) to obtain a dominant map $T^2 \times \Sigma_k \rightarrow N$. This map is not a branched covering, not even in the case where N is diffeomorphic to $\#_k \mathbb{C}P^2 \#_k \overline{\mathbb{C}P^2}$, because the branch locus of (3) intersects the preimage of the branch locus of (1) in a codimension four subset of $S^1 \times (\#_k S^2 \times S^1)$.

Nevertheless, we note that $T^2 \times \Sigma_k$ is a branched four-fold cover of $\#_k(S^2 \times S^2)$, by Remark 2. The existence of dominant maps from products $T^2 \times \Sigma_k$ to every simply connected four-manifold has also been shown by Kotschick and Löh [15], using a result of Duan and Wang [8]. That result states that if X and Y are oriented, closed four-manifolds and Y is simply connected, then a degree $d \neq 0$ map $f: X \rightarrow Y$ exists if and only if the intersection form of Y , multiplied by d , is embedded into the intersection form of X , where the embedding is given by $H^*(f)$ (the ‘‘only if’’ part is obvious).

As we mention in Section 3, the above existence criterion by Duan and Wang implies that every integer can be realized as the degree of a map from T^4 to $\#_3(S^2 \times S^2)$, although no such map can be deformed to a branched covering by a recent result of Pankka and Souto [19]. However, according to Theorem 4, it is natural to ask when a π_1 -surjective non-zero degree map between two connected, oriented, closed four-manifolds is homotopic to the composition of a branched covering with a pinch map. Another natural question in this context is the following (suggested to me by J. Souto): Let N be a connected, oriented, closed four-manifold and $f: M \rightarrow N$ a π_1 -surjective map of non-zero degree. Is there a $k \geq 0$ so that the composition of the pinch map $\#_k(S^2 \times S^2) \# M \rightarrow M$ with f is homotopic to a branched covering?

5.3. Domination by products and sets of self-mapping degrees. Kotschick and Löh [15] asked whether every closed manifold with finite fundamental group is dominated by a product. In this paper, we have answered in positive that question in dimensions four and five and in certain higher dimensions. For all our dominant maps onto a simply connected n -dimensional manifold, the domain can be taken to be a product of the circle with a connected sum of copies of T^{n-1} . According

to this, one could ask whether every closed manifold with finite fundamental group is dominated by a product of type $S^1 \times N$, where N is an $(n - 1)$ -dimensional manifold (e.g. $N = \#_k T^{n-1}$ for some $k \geq 0$), or more generally, by a product of type $S^m \times N$. This is a considerably stronger question than that of [15], however, it seems less likely to be true. For instance, simply connected manifolds which admit self-maps of absolute degree at most one might not be dominated by products (at least) of type $S^m \times N$. Such manifolds exist and are called inflexible; see for example [4]. In this direction, and since the set of self-mapping degrees of $S^m \times N$ is infinite, we ask the following (at least in the simply connected case): Are there examples of inflexible manifolds that are dominated by flexible ones (i.e. by manifolds whose sets of self-mapping degrees are infinite)?

We note that, if every simply connected, closed n -dimensional manifold is dominated by a product $X_1 \times X_2$ such that one of the factors X_i has infinite set of self-mapping degrees, then every finite functorial semi-norm in degree n vanishes on simply connected manifolds, which is an open question by Gromov [13, Chapter 5G₊].

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