SAMPLE PROBLEMS: MATH 323 EXAMS
SPRING 2021

These are all problems from previous exams, or problems that would be appropriate for future exams. Solutions will not be supplied by the instructor, though the problems would make good topics for office hours. **The best way to use this document is to use each problem as an example of a problem type, and to then make up your own problems.** The best (and probably the only) way to be sure you're ready for a test is to verify that you can create and solve your own test problems.

Please refer to the tentative schedule for the list of sections covered by each exam.

Chapter 12

12.1.

(1) Suppose you're weightless, floating around in $\mathbb{R}^3$. You notice that from your perspective the positive x axis is pointing to your left and the positive y axis is pointing down. Which way should the z axis be pointing?
(2) Now draw the axes in the standard way, with x pointing to the right and z pointing up. Plot a point in $\mathbb{R}^3$ that shows where your feet might be in the above situation, and draw an arrow from that point toward your head.
(3) If you're one unit tall, write coordinates for the head and foot of the arrow.
(4) How far are your feet from the origin?
(5) Write the equation for the sphere which is centered at your feet, and such that the origin is a point on the sphere.

12.2. Choose a pair of vectors $\mathbf{u}, \mathbf{v}$ in $\mathbb{R}^2$. This means actually choosing four random numbers.

(1) Make sure $\mathbf{u}$ and $\mathbf{v}$ are not parallel.
(2) Draw the relevant triangles, parallelograms, etc. for the following:
  (a) $\mathbf{u} + \mathbf{v}$
  (b) $\mathbf{u} - \mathbf{v}$
  (c) $2\mathbf{u}$
  (d) $-\mathbf{u}$
(3) Explain to yourself how to do the same thing for vectors in $\mathbb{R}^3$.
(4) Find a formula for the vector whose head is at one point in $\mathbb{R}^3$ and whose tail is at another point of $\mathbb{R}^3$.
(5) Why is the distance between two points (x_1, y_1, z_1) and (x_2, y_2, z_2) the same as the magnitude of the vector $\langle x_2 - x_1, y_2 - y_1, z_2 - z_1\rangle$?

12.3. Choose a pair of vectors $\mathbf{u}, \mathbf{v}$ in $\mathbb{R}^3$. This means actually choosing six random numbers.

(1) Calculate $\mathbf{u} \cdot \mathbf{v}$.
(2) What does it mean geometrically if the answer above is 0?
(3) Find the cosine of the angle between $\mathbf{u}$ and $\mathbf{v}$.

(4) Find $\text{proj}_{\mathbf{v}}\mathbf{u}$, $\text{proj}_{\mathbf{u}}\mathbf{v}$, $\text{comp}_{\mathbf{v}}\mathbf{u}$ and $\text{comp}_{\mathbf{u}}\mathbf{v}$. Draw pictures.

12.4. Choose three vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in $\mathbb{R}^3$. This means actually choosing nine random numbers.

(1) Calculate $\mathbf{u} \times \mathbf{v}$. What does it mean if the answer is $\mathbf{0}$?
(2) Lay a clock on the table. If the vector $\mathbf{a}$ points toward 12 o'clock and the vector $\mathbf{b}$ points at 11 o'clock, which direction does $\mathbf{a} \times \mathbf{b}$ point?
(3) If you already knew what $\mathbf{w} \times \mathbf{v}$ was, could you easily imagine $\mathbf{v} \times \mathbf{w}$?
(4) Find the sine of the angle between $\mathbf{u}$ and $\mathbf{v}$.
(5) Find the area of a parallelogram which has two sides $\mathbf{u}$, $\mathbf{v}$.
(6) Find the volume of a parallelepiped which has three sides $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$.
(7) Find the torque given by applying the force $\mathbf{F} = \mathbf{u}$ to the end of the torque arm $\mathbf{r} = \mathbf{v}$.

12.5. Choose four points A, B, C, D in $\mathbb{R}^3$. This means actually choosing 12 random numbers.

(1) Make sure that these points do not lie on a single line.
(2) Find the vector and scalar equations for the plane containing A, B, C.
(3) Find the vector and parametric equations for the line containing A, B.
(4) Write two vector equations for lines in R^3. Find the point(s) at which these lines intersect, if any.
(5) Same for a pair of planes.
(6) Give two lines which intersect orthogonally at A.
(7) Give two planes which intersect orthogonally along the line

$$\mathbf{r}(t) = \langle 2 + t, 3 - 2t, 1 \rangle.$$

(8) How do you quickly figure out whether two lines or two planes are parallel?
(9) Find the distance from D to the plane from (2).
Hint: The vector pointing from D to the nearest point on the plane is orthogonal to the plane.

12.6.

(1) Write a quadratic equation in the variables x, y, z and sketch three traces.
(2) Explain to someone else what a trace represents geometrically. You might consider trails which appear to be straight North/South or East/West in a map of a mountainous region.
(3) Explain why the graph of an equation like $y = x$ could be a line or a plane, depending on context.
(4) Write equations for a sphere, a paraboloid, and a cylinder. What would you change to increase the radius of the sphere or the cylinder? How would you move the tip of the paraboloid to a different point of $\mathbb{R}^3$, or make the paraboloid open wider?

Chapter 13

13.1.

(1) Find all points of intersection between the curve $\mathbf{r}(t) = 3t\mathbf{i} - t\mathbf{k}$ and the paraboloid $x^2 + z^2 = y$.

(2) Find a vector function that represents the curve of intersection of the cylinder $x^2 + z^2 = 4$ and the plane $y + x = 4$.
Hint: Let $x = 2\cos t$, $y = 2\sin t$.

(3) Evaluate

$$\lim_{t\to 0} \left\langle \frac{\sin t}{t}, t\ln t, t+2. \right\rangle$$

(4)

13.2.

(1) Find a unit tangent vector to the graph of the following function at $\mathbf{r}(\pi)$.

$$\mathbf{r}(t) = \left\langle \frac{\sin t}{t}, t\ln t, t+2. \right\rangle$$

(2) Evaluate

$$\int_0^{\pi/4} \left\langle 3e^t, \tan t, 5 \right\rangle dt$$

(3) Find an equation for the tangent line to the curve at the point p.

$$\mathbf{r}(t) = \left\langle e^{-t\cos(\pi t)}, 2^t, \sin^2(\pi t) \right\rangle, \qquad p = (e, 2, 0).$$

13.3.

(1) A particle moves along the space curve given by

$$\mathbf{r}(t) = \langle t, \cos(2\pi t), \sin(2\pi t) \rangle, \ t \in [0, 2].$$

Find the length of the curve.

(2) A maple seed spins through the air like a helicopter along the path given by

$$\mathbf{r}(t) = \left\langle t, 5, \frac{1}{3}\left(t^2+2\right)^{3/2} \right\rangle, \qquad t \in [0, 2].$$

If distance is in meters, what is the length of the path it traced?

13.4. A particle moves along the space curve given by

$$\mathbf{r}(t) = \left\langle t^2, 3, e^{-t} \right\rangle, \ t \in [0, 2].$$

(1) How fast was the particle moving at time $t = 1$?
(2) What direction was it moving at time $t = 1$?
(3) Was the particle speeding up or slowing down at time $t = 1$?

Chapter 14

14.1.

(1) Sketch the domain of the function $f(x, y) = \ln(x) - \sqrt{y} + \frac{1}{x+y}$.
(2) Sketch the domain of the following function.

$$f(x, y) = \frac{\sqrt{4 - x^2 - y^2}}{\ln(x - y)}$$

(3) Explain what the level curves of a function of two variables tells you about the shape of the graph of the function.

(4) If you wanted to move a point (x, y) in the direction that increases $f(x, y)$ the fastest, would that direction be parallel or perpendicular to nearby level curves?

(5) If a point is moving along a level curve of $f(x, y)$, what is happening to the z coordinate of the point on the graph above/below that point?

14.2.

(1) Find the limit if it exists, or show it does not exist.

(a)
$$\lim_{(x,y)\to(0,0)} \frac{x^4 - 14y^2}{x^2 + 7y^2}$$

(b)
$$\lim_{(x,y)\to(0,0)} \frac{x^3 - y^3}{x^2 + xy + y^2}$$

(c)
$$\lim_{(x,y)\to(1,0)} \frac{xy - y}{(x-1)^2 + y^2}$$

(d)
$$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2 + y^2}}$$

14.3.

(1) Suppose $f_x(2, 2) = 2$. What does this imply about the graph of f?

(2) Find the first partial derivatives of $f(x, y) = y \sin(xy)$.

(3) Find the first partial derivatives of $f(x, y) = \int_x^y t^2 dt$.

(4) Find $f_{xy}(1/3, \pi)$ and $f_{yx}(1/3, \pi)$ for the function $f(x, y) = xy \cos(x + y)$.

14.4.

(1) Write an equation for the plane which is tangent to the graph of $f(x, y) = x^2 - y^3$ at the point $(-1, -1, 2)$.

(2) Is the plane $z = -2(x+1) - 3(y-1)$ tangent to the graph of $f(x, y) = x^2 - y^3$ at some point? Remember the formula you used to find the answer to the previous question.

(3) Use the linear approximation of $f(x, y) = x^2 - y^3$ at $(1, 2)$ to estimate $f(0, 0)$.

14.5.

(1) Draw the dependence tree and use the chain rule from class to find $\frac{\partial P}{\partial x}$ at the point $(x, y) = (1, 3)$, where
$$P = b^2 a - \sin(ac), \ a = xy, \ b = ye^{2x}, \ c = x^2.$$

(2) Suppose
$$\begin{aligned} f(x, y, z) &= x^2 e^{yz} \\ x(a, b) &= 3a + 2b \\ y(a, b) &= \cos(a - b) \\ z(a, b) &= ab. \end{aligned}$$

Find $\frac{\partial f}{\partial a}$ at $(a,b) = (2,2)$.

Suppose $a = 3t + 2$ and $b = 5s$. Would that change your answer?

14.6.

(1) Find the directional derivative of $f(x,y) = \sin(x+y)$ at the point $(0, \pi/3)$ in the direction of $\langle -1, 3 \rangle$.
(2) Explain what your answer says about the graph of f.
(3) Calculate $\nabla f\left(\frac{\pi}{2}, \frac{\pi}{3}\right)$ and explain what your answer says about the graph of f.
(4) What would it mean for the graph of f if the gradient of f vanishes at a point?
(5) What would it mean for the graph of a function $f(x,y)$ if $D_{\mathbf{u}} f(2,2) = 0$?
(6) Suppose the temperature at (x,y,z) is given by $f(x,y,z) = xe^{yz}$.
    (a) At the origin, in which direction is the temperature increasing the fastest?
    (b) At the origin, is there a direction in which the temperature is neither increasing nor decreasing? If so, find one such direction.

14.7.

(1) Find and classify all critical points for the following function as either local maximum, local minimum, or saddle points.

$$f(x,y) = 2 - \frac{1}{4}x^4 + 2x^2 - y^2$$

(2) Find the absolute maximum and minimum values of f over the rectangle $[-1,1] \times [-2,2]$.

14.8.

(1) Use the method of Lagrange multipliers to find the distance from the origin to the plane given by $-2(x-1) + (y+1) + 3z = 0$.
(2) Find the absolute maximum and minimum values of $f(x,y) = e^{-xy}$ subject to the constraint that $x^2 + 2y^2 = 3$.
(3) Explain the steps for a Lagrange multipliers problem to a sock puppet decorated with stick-on googly eyes, pipe cleaners and glitter glue.

Chapter 15

15.1.

(1) Evaluate $\int_0^3 \int_1^2 x^2 - y^2 dy dx$.
(2) The following integrals give the volume of various solid shapes. Describe those shapes in words.

$$\int_R 5 \; dA, \;\; R: \;\; x \in [-2,6], \;\; y \in [1,5]$$

$$\int_R x + y \; dA, \;\; R = [-1,1] \times [-1,1]$$

15.2.

(1) Let R be the triangle in the xy plane bounded by the lines $y = x + 3$, $y = 4 - x$, and $x = -1$. Find the volume of the solid between R and the graph of $f(x, y) = x^2 y$.
(2) Write an integral that expresses the volume of the solid between the graph of $f(x, y) = ye^x$ and the region in the xy plane satisfying $x^2 + \frac{1}{4}y^2 \leq 1$.
(3) Sketch the region of integration and switch the order of integration for the following integral:

$$\int_0^2 \int_0^{\sqrt{4-x^2}} f(x, y)\ dydx$$

(4) Listen to a classmate explain the difference between a type I and a type II region in the plane without interrupting. When they're finished, give them an example of a type II region, with appropriate limits of integration.

15.3.

(1) Let D be the top half of the disk in $\mathbb{R}^2$ whose center is the origin and whose radius is 4. By changing to polar coordinates, evaluate the following integral:

$$\iint_D 4x^2 y\ dA$$

(2) Raise the graph of $z = x^2 + y^2$ by one unit above the xy plane and consider the solid between this surface and the unit disk in the xy plane centered at the origin. Write and evaluate a double integral in polar coordinates to find the volume of this solid.

15.6.

(1) Set up the integral $\iiint_E y^2 dV$ in rectangular coordinates, where E is the solid tetrahedron bounded by the coordinate planes and the plane $2x+6y+3z = 6$. No need to evaluate the integral.
(2) Evaluate the triple integral $\iiint_E 6xy\ dV$, where E lies under the plane $z = 1 + x + y$ and above the region in the xy plane bounded by the curves $y = \sqrt{x}$, $y = 0$, and $x = 1$.
(3) Express the integral $\iiint_E f(x, y, z)\ dV$ using all six orders of integration, where E is the solid in the first octant bounded by the coordinate planes and the graph of $z = 9 - x^2 - y^2$.

15.7.

(1) Change $(-1, 1, 1)$ from rectangular to cylindrical coordinates.
(2) Change $(4, \pi/3, -2)$ from cylindrical to rectangular coordinates.
(3) Write the integral of $f(x, y, z)$ over the region between the paraboloid $z = x^2 + y^2$ and the plane $z = 9$.
(4) Find the volume of the region bounded by the cone $z = \sqrt{x^2 + y^2}$ and the upper hemisphere of the sphere with radius 3.
(5) A cylindrical tower has radius R meters and a roof which is a hemisphere. The total height of the tower with its roof is $R + H$ meters.

(a) Sketch the tower and label its various measurements using cylindrical coordinates. Be sure to include a function $z = f(r, \theta)$ whose graph is the top of the tower. Be careful to make sure that f does not describe a dome that is sitting on the ground.
(b) Write an integral in cylindrical coordinates to express the volume of the tower.
(c) Calculate the volume of the tower one more time, this time using the formula $\pi R^2 H$ for the volume of a cylinder and $\frac{4}{3}\pi R^3$ for the volume of a ball. Do your answers match?

15.8.

(1) Give equations or inequalities in spherical coordinates for the part of the unit sphere centered at the origin that lies in the first octant. What changes if it's the unit ball instead?
(2) What does the graph of $\rho = \sec\phi$ look like?
(3) Give an equations or inequalities in spherical coordinates for a hollow ball with outer diameter 30 cm and thickness 0.5 cm.
(4) Find the volume of the region bounded by the cone $z = \sqrt{x^2 + y^2}$ and the upper hemisphere of the sphere with radius 3.
(5) Evaluate

$$\iiint_B \left(x^2 + y^2 + z^2\right)^2 \; dV,$$

where B is the ball with radius 5 centered at the origin.

Chapter 16

16.1.

(1) Explain what information a plot of ∇f would give about the graph of f.
(2) Plot enough of the following vector fields to get a good idea of what they look like.
  (a) ∇f, where $f(x, y) = x^2 - y^2$
  (b) $\langle 0, y\rangle$
  (c) $\langle 1, y\rangle$
  (d) $\langle x, y\rangle$

16.2.

(1) Let C be the linear path from $A(0, 0, 0)$ to $B(1, -2, 3)$ to $C(0, -1, 1)$ (so C is given by two line segments) and let $f(x, y, z) = y^2 - 2xz$.
  (a) Find $\int_C f \; ds$.
  (b) Find $\int_C f \; dy$.
(2) Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = \left\langle x^2 y, -3y\right\rangle$ and C is given by the function $\mathbf{r}(t) = \langle \cos t, \sin t\rangle$, $t \in [0, \pi/2]$.

16.3.

(1) Is the vector field $\langle ye^x, e^x + e^y\rangle$ conservative?
(2) Determine whether the following line integral is independent of path and evaluate the integral.

$$\int_C \cos y \; dx - x \sin y \; dy,$$

where C is any path from $(0,\pi)$ to $(3,\pi/2)$.

(3) Evaluate $\int\limits_0^1 \nabla f \cdot d\mathbf{r}$, where

$$f(x,y,z) = e^{\arctan^2(xyz)}, \qquad \mathbf{r}(t) = \left\langle t, \ln((e-1)t+1), \sin\left(\frac{\pi}{2}t\right)\right\rangle.$$

(4) Use the fundamental theorem of line integrals to calculate

$$\int_C \left\langle y^2z + 2xz^2, 2xyz, xy^2 + 2x^2z\right\rangle \cdot d\mathbf{r}$$

where C is the curve $\langle x,y,z\rangle = \left\langle \sqrt{t}, t+1, t^2\right\rangle$ from $t=0$ to $t=1$.

16.4.

(1) Recite Green's theorem from memory. Explain how to convert the integrals on each side of the equation into regular old definite integrals of real-valued functions.

(2) Use Green's theorem to evaluate $\int_C 2e^x\,dx + y^2\,dy$, where C is the rectangle forming the positively-oriented boundary of $[-1,1]\times[2,3]$.

(3) Use Green's theorem to evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F}(x,y) = \left\langle y^3, -x^3\right\rangle$ and C is the circle with radius 2 centered at $(0,0)$ oriented clockwise.

(4) Use Green's theorem to compute

$$\int_C \langle ye^x, e^x + x\rangle \cdot\ d\mathbf{r}$$

where C is the counter-clockwise triangle with vertices $(0,0)$, $(3,0)$ and $(0,1)$.

16.5.

(1) Find the curl and divergence of the vector field

$$\mathbf{F}(x,y,z) = \left\langle xye^z, \cos(y^2), x\right\rangle.$$

(2) Let f be a scalar field and $\mathbf{F}$ a vector field. State whether each expression is meaningful. If not, explain why. If so, state whether it is a scalar field or a vector field.

(a) $\operatorname{curl} f$	(b) $\operatorname{grad} f$	(c) $\operatorname{div}\mathbf{F}$
(d) $\operatorname{curl}(\operatorname{grad} f)$	(e) $\operatorname{grad}\mathbf{F}$	(f) $\operatorname{grad}(\operatorname{div}\mathbf{F})$
(g) $\operatorname{div}(\operatorname{grad} f)$	(h) $\operatorname{grad}(\operatorname{div} f)$	(i) $\operatorname{curl}(\operatorname{curl}\mathbf{F})$
(j) $\operatorname{div}(\operatorname{div}\mathbf{F})$	(k) $(\operatorname{grad} f)\times(\operatorname{div}\mathbf{F})$	(l) $\operatorname{div}(\operatorname{curl}(\operatorname{grad} f))$

(3) How do you use the curl operator to check whether a vector field is conservative?

16.6.

(1) Does the point $(1,0,3)$ or the point $(-1,0,1)$ lie on the parametric surface given by $\left\langle u-v, u^2, v\right\rangle$?

(2) Find an equation for the tangent plane to the surface given by $\left\langle u^3, 3-uv, e^{uv}\right\rangle$ at the point $(0,3,1)$.

(3) Find the area of the surface $z = \frac{2}{3}\left(x^{3/2} + y^{3/2}\right)$, where $(x,y)\in[0,1]\times[0,1]$.

(4) Find the surface area of S, where S is parameterized by

$$\mathbf{r}(u,v) = \langle u, u-v, u+v\rangle$$

with (u,v) in the region bounded by the curves $v = u^2$ and $v = 1 - u^2$.

16.7.

(1) Explain what the difference is between dS and $d\mathbf{S}$ a surface integral.
(2) Evaluate $\iint_S x^2 + y^2 \ dS$, where S is the surface with vector equation $\mathbf{r}(u,v) = \left\langle 2uv, u^2 - v^2, u^2 + v^2\right\rangle$, $u^2 + v^2 \leq 1$.
(3) Evaluate the surface integral

$$\iint_S \mathbf{F} \cdot \ d\mathbf{S}$$

where S is the surface defined by $z = x^2 + y^2$ subject to $x^2 + y^2 \leq 1$ and $\mathbf{F}(x,y,z) = \langle x, y, z\rangle$.

16.8.

(1) Recite Stokes' Theorem from memory. Explain how to convert the integrals on each side of the equation into regular old definite integrals of real-valued functions.
(2) Use Stokes' theorem to calculate $\iint_S \operatorname{curl}\mathbf{F} \cdot \ d\mathbf{S}$, where

$$\mathbf{F}(x,y,z) = \left\langle 3x, 2y^2, e^{x+z}\right\rangle$$

and S is a surface whose positively-oriented boundary is the circle given by $\langle 3, 2\sin t, 2\cos t\rangle$, $t \in [0, 2\pi]$.
(3) Use Stokes' theorem to calculate $\iint_S \operatorname{curl}\mathbf{F} \cdot \ d\mathbf{S}$, where

$$\mathbf{F} = \left\langle x^2 \sin z, y^2, -xze^{cosy}\right\rangle$$

and S is the part of the paraboloid $z = 4 - x^2 - y^2$ that lies above the xy plane, oriented downward.

16.9.

(1) Recite the Divergence Theorem from memory. Explain how to convert the integrals on each side of the equation into regular old definite integrals of real-valued functions.
(2) Use the Divergence Theorem to calculate the flux $\iint_S \mathbf{F} \cdot \ d\mathbf{S}$, where $\mathbf{F} = \left\langle 2x^3 + y^3, y^3 + z^3, 3y^2 z\right\rangle$ and S is the surface of the solid bounded by the paraboloid $z = 1 - x^2 - y^2$ and the xy plane.
(3) Verify that the Divergence Theorem is true for the vector field $\mathbf{F}(x,y,z) = \langle 1, x, y\rangle$ on any solid cube centered at the origin.