To all 323 students:

The following are two old midterm exams (titled “Sample 1” & “Sample 3”) that were given for Math 323 Exam 1. Please ignore the following problems from these exams:

-First Test “Sample 1”: #3, #6, #8

-Second Test “Sample 3”: #4, #5, #6(c), #7(a), #8(a)

The problems above involve topics which will not be covered on your test.

Math 323 Midterm Examination 1, Sample 1

Problem 1. The line L is given by the equation $(x, y, z) = (2+t, 1-t, 3t)$, and the plane X is given by the equation $2x+3y+z = 11$. Find the following:

a) the point at which L intersects X.

b) the equation of the plane that contains L and is perpendicular to X.

Problem 2. Find all points on the curve $\vec{r}(t) =< 2t, t^2, t^3 >$, where the tangent line is parallel to the plane

$$-6x + 3y + 2z = 15.$$

Problem 3. Find the curvature at the point $(0, 0, 1)$ of the curve

$$\vec{r}(t) =< t, t^2, e^t > .$$

Problem 4. Find all moments t, when the acceleration vector is perpendicular to the velocity vector for

$$\vec{r}(t) =< t^2, \ln t, 2t + 1 >$$

Problem 5. a) Draw the parametric curve

$$\vec{r}(t) =< \cos t, \sin t, t >, \qquad 0 \leq t \leq 4\pi$$

b) Find its length.

Problem 6. Find the unit normal vector and the unit binormal vector at the point $(1, \frac{2}{3}, 3)$ for the curve $\vec{r}(t) =< t^2, \frac{2}{3}t^3, t + 2 >$.

Problem 7. Find the equations of the parallel planes that contain the skew lines $\overline{AB}$ and $\overline{CD}$, where

$$A = (1, 0, 2),\ B = (2, 3, -1),\ C = (2, 4, 0),\ D = (3, 4, 1)$$

Problem 8. Find the normal and the tangential components of the acceleration for the curve

$$\vec{r}(t) = < t + 1, t^2 - 1, t^3 >$$

at the point $(3, 3, 8)$.

Math 323 Midterm Examination 1, Sample 3

Problem 1. Find the equation of the plane containing the points $A = (1, 2, 3)$, $B = (2, 0, 1)$, and $C = (3, 2, 1)$.

Problem 2. a) Find the center and the radius of the sphere

$$x^2 + y^2 + z^2 - 4x + 6z = 4$$

b) Draw this sphere in the coordinates.

Problem 3. For the vectors $\vec{u} =< 2, 1, 4 >$ and $\vec{v} =< 3, 0, 1 >$ find
a) $comp_{\vec{u}}\vec{v}$
b) $proj_{\vec{u}}\vec{v}$

Problem 4. Find the distance between the x-axis and the tangent line to the curve $(x, y, z) = (2t - 1, t^2 + 1, t^3)$ at the point $(1, 2, 1)$

Problem 5. Find all points on the curve $\vec{r}(t) =< 2t^2, t^3 + 3t, t^2 + 1 >$ where the curvature is zero.

Problem 6. a) Set up the integral for the length of the curve

$$\vec{r}(t) =< t, t^2, t^3 >$$

between the point $(1, 1, 1)$ and the point $(-1, 1, -1)$.

b) Find the unit tangent vector at the point $(-1, 1, -1)$.

c) Find the normal and tangential components of the acceleration at the point $(-1, 1, -1)$.

Problem 7. a) Reparametrize the curve

$$(x, y, z) = (\sin t, \cos t, \tan t), \quad 0 \leq t < \frac{\pi}{3}$$

using the new variable $u = \tan t$. Simplify your answer.

b) For the new parametrization, find the acceleration at the point $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 1)$.

Problem 8. a) Find the distance between the point $P = (1, 2, 3)$ and the plane $x + y - 2z = 8$.

b) Find the parametric **and** the symmetric equations of the perpendicular from P to that plane.

c) Find the cooridnates of the point at which the perpendicular intersects the plane.

Math 323 Midterm Examination 1, Sample 1

Problem 1. The line L is given by the equation $(x, y, z) = (2+t, 1-t, 3t)$, and the plane X is given by the equation $2x+3y+z = 11$. Find the following:

a) the point at which L intersects X.

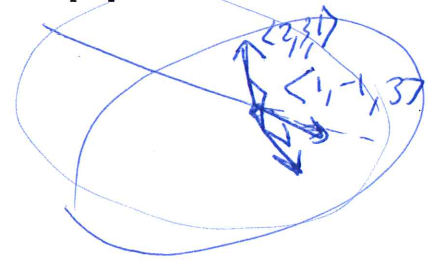
b) the equation of the plane that contains L and is perpendicular to X.

$2(2+t)+3(1-t)+3t=11$

$4+2t+3-3t+3t=11$

$t=2$

a) $P=(4,-1,6)$



$\langle 1, -1, 3\rangle \times \langle 2, 3, 1\rangle = \langle -10, 5, 5\rangle = 5\cdot\langle -2, 1, 1\rangle$

b) So equation is:

$-2(x-4)+1(y+1)+1\cdot(z-6)=0$

$-2x+y+z=-3$

(Alternatively, $(2,1,0)$ point can be used)

Problem 2. Find all points on the curve $\vec{r}(t) = \langle 2t, t^2, t^3\rangle$, where the tangent line is parallel to the plane

$$-6x + 3y + 2z = 15.$$

$\vec{v}(t) = \langle 2, 2t, 3t^2\rangle$

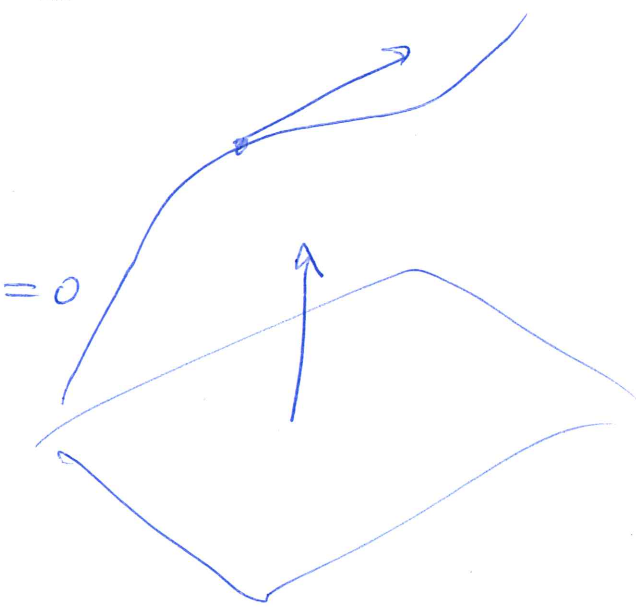
$\vec{v}(t) \perp \langle -6, 3, 2\rangle$

$2\cdot(-6)+2t\cdot 3+3t^2\cdot 2=0$

$-12+6t+6t^2=0$

$t^2+t-2=0$

$t=1,-2$



$t=1$: $(2,1,1)$

$t=-2$: $(-4,4,-8)$

Note: both points are not on the plane, so the tangent lines are parallel, not on the plane

Problem 3. Find the curvature at the point $(0, 0, 1)$ of the curve

$t=0$

$$\vec{r}(t) = < t, t^2, e^t > .$$

$r' = \langle 1, 2t, e^t \rangle \overset{t=0}{=} \langle 1, 0, 1 \rangle$

$r'' = \langle 0, 2, e^t \rangle \overset{t=0}{=} \langle 0, 2, 1 \rangle$

$r' \times r'' = \langle -2, -1, 2 \rangle$

$$\kappa = \frac{|r' \times r''|}{|r'|^3} \overset{t=0}{=} \frac{\sqrt{4+1+4}}{(\sqrt{2})^3} = \boxed{\frac{3}{2\sqrt{2}}}$$

Problem 4. Find all moments t, when the acceleration vector is perpendicular to the velocity vector for

$$\vec{r}(t) = < t^2, \ln t, 2t + 1 >$$

$r' = \langle 2t, \frac{1}{t}, 2 \rangle$

$r'' = \langle 2, -\frac{1}{t^2}, 0 \rangle$

$0 = r' \cdot r'' = 4t - \frac{1}{t^3}$

So $4t = \frac{1}{t^3}$,

$t^4 = \frac{1}{4}$

$t = \pm \frac{1}{\sqrt{2}}$

but: $t > 0$

So $t = \frac{1}{\sqrt{2}}$

The point is

$$\boxed{\langle \frac{1}{2}, \ln\left(\frac{1}{\sqrt{2}}\right), \sqrt{2}+1 \rangle}$$

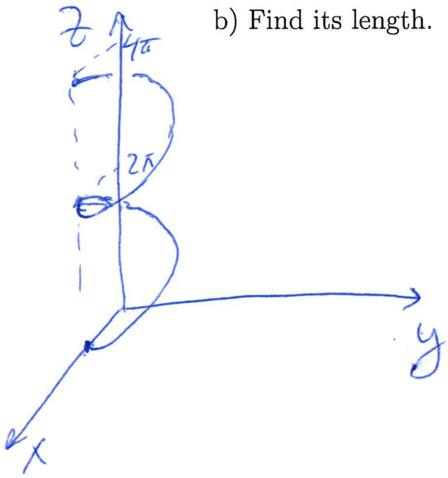
$\ln\left(\frac{1}{\sqrt{2}}\right) = \left(-\frac{1}{2}\ln 2\right)$

This simplification is optional

Problem 5. a) Draw the parametric curve

$$\vec{r}(t) = < \cos t, \sin t, t >, \qquad 0 \le t \le 4\pi$$ $\vec{v}(t) = \langle -\sin t, \cos t, 1 \rangle$

b) Find its length.



$$L = \int_0^{4\pi} \sqrt{\sin^2 t + \cos^2 t + 1}\, dt = \int_0^{4\pi} \sqrt{2}\, dt = \boxed{4\sqrt{2}\pi}$$

Problem 6. Find the unit normal vector and the unit binormal vector at the point $(1, \frac{2}{3}, 3)$ for the curve $\vec{r}(t) = < t^2, \frac{2}{3}t^3, t+2 >$.

$$\begin{cases} t^2 = 1 \\ \frac{2}{3}t^3 = \frac{2}{3} \\ t+2 = 3 \end{cases} \Rightarrow t = 1$$

$r' = \langle 2t, 2t^2, 1 \rangle \overset{t=1}{=} \langle 2, 2, 1 \rangle$

$r'' = \langle 2, 4t, 0 \rangle \overset{t=1}{=} \langle 2, 4, 0 \rangle$

$\vec{B}$ is normalized $r' \times r''$ $\langle -4, 2, 4 \rangle$

So $\vec{B} = \dfrac{\langle -4, 2, 4 \rangle}{\sqrt{16+4+16}} = \langle -\frac{2}{3}, \frac{1}{3}, \frac{2}{3} \rangle$

$\vec{N}$ is normalized $\vec{B} \times \vec{r}'$

$3\vec{B} \times \vec{r}' = \langle -2, 1, 2 \rangle \times \langle 2, 2, 1 \rangle = \langle -3, 6, -6 \rangle$

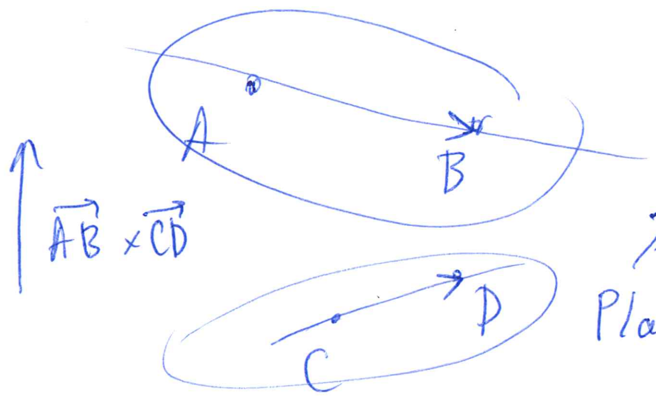
$$\begin{array}{r} \langle -2\,1\,2 \rangle \\ \times\ \langle 2\,2\,1 \rangle \\ \hline \langle -3, 6, -6 \rangle \end{array}$$

So $\boxed{\vec{N} = \langle -\frac{1}{3}, \frac{2}{3}, -\frac{2}{3} \rangle}$

(By the way, $\vec{T} = \langle \frac{2}{3}, \frac{2}{3}, \frac{1}{3} \rangle$)

Problem 7. Find the equations of the parallel planes that contain the skew lines $\overline{AB}$ and $\overline{CD}$, where

$$A = (1, 0, 2),\ B = (2, 3, -1),\ C = (2, 4, 0),\ D = (3, 4, 1)$$



$\overrightarrow{AB} = \langle 1, 3, -3 \rangle$

$\overrightarrow{CD} = \langle 1, 0, 1 \rangle$

$\overrightarrow{AB} \times \overrightarrow{CD} = \langle 3, -4, -3 \rangle$

Plane through A: $3(x-1) - 4y - 3(z-2) = 0$

$3x - 4y - 3z = -3$

Plane through C: $3x - 4y - 3z = -10$

Problem 8. Find the normal and the tangential components of the acceleration for the curve

$$\vec{r}(t) = < t + 1, t^2 - 1, t^3 >$$

at the point $(3, 3, 8)$. $t = 2$

$r' = \langle 1, 2t, 3t^2 \rangle$, $r'' = \langle 0, 2, 6t \rangle$

At $t = 2$: $r' = \langle 1, 4, 12 \rangle$, $r'' = \langle 0, 2, 12 \rangle$

Tangential component: $\dfrac{r' \cdot r''}{|r'|} = \dfrac{0 + 8 + 144}{\sqrt{1 + 16 + 144}} = \dfrac{152}{\sqrt{161}}$

Normal component: $\dfrac{|r' \times r''|}{|r'|} = \dfrac{2 \cdot \sqrt{12^2 + 36 + 1}}{\sqrt{1 + 16 + 144}} = \dfrac{2 \cdot \sqrt{181}}{\sqrt{161}}$

$r' = \langle 1, 4, 12 \rangle$

$r'' = \langle 0, 2, 12 \rangle$

$r' \times r'' = \langle 24, -12, 2 \rangle$

$= 2 \langle 12, -6, 1 \rangle$

Math 323 Midterm Examination 1, Sample 3

Problem 1. Find the equation of the plane containing the points $A = (1, 2, 3)$, $B = (2, 0, 1)$, and $C = (3, 2, 1)$.

$\vec{AB} = \langle 1, -2, -2 \rangle$

$\vec{AC} = \langle 2, 0, -2 \rangle$

$\vec{AB} \times \vec{AC} = \langle 4, -2, 4 \rangle$

$4(x-1) - 2(y-2) + 4(z-3) = 0$

$4x - 2y + 4z = 12$

Or $\boxed{2x - y + 2z = 6}$

Problem 2. a) Find the center and the radius of the sphere

$$x^2 + y^2 + z^2 - 4x + 6z = 4$$

b) Draw this sphere in the coordinates.

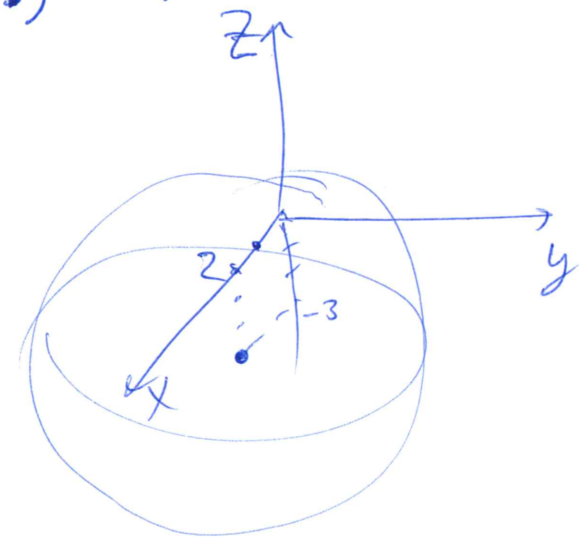
a) $(x^2 - 4x) + y^2 + (z^2 + 6z) = 4$

$(x^2 - 4x + 4) + y^2 + (z^2 + 6z + 9) = 4 + 4 + 9$

$(x-2)^2 + y^2 + (z+3)^2 = 17$

Center $(2, 0, -3)$

Radius $\sqrt{17}$



Problem 3. For the vectors $\vec{u} =< 2, 1, 4 >$ and $\vec{v} =< 3, 0, 1 >$ find
a) $comp_{\vec{u}}\vec{v}$
b) $proj_{\vec{u}}\vec{v}$

a) $comp_{\vec{u}} \vec{v} = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}|} = \frac{10}{\sqrt{21}}$

b) $proj_{\vec{u}} \vec{v} = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}|^2} \cdot \vec{u} = \frac{10}{21}\langle 2,1,4\rangle = \left\langle \frac{20}{21}, \frac{10}{21}, \frac{40}{21} \right\rangle$

Problem 4. Find the distance between the x-axis and the tangent line to the curve $(x, y, z) = (2t - 1, t^2 + 1, t^3)$ at the point $(1, 2, 1)$ $\quad t = 1$

$\vec{v} = \langle 2, 2t, 3t^2 \rangle \quad \langle 2, 2, 3 \rangle$ at $t = 1$

Line: $(x, y, z) = \langle 1 + 2\tau, 2 + 2\tau, 1 + 3\tau \rangle$

Square of the distance to x-axis: $(2+2\tau)^2 + (1+3\tau)^2 = d^2$

$d^2 = 13\tau^2 + 14\tau + 5$

Minimum at $\tau = -\frac{7}{13}$.

$d^2 = 13 \cdot \left(-\frac{7}{13}\right)^2 + 14\left(-\frac{7}{13}\right) + 5 = -\frac{49}{13} + 5 = \frac{16}{13}$

$d = \frac{4}{\sqrt{13}}$

* See also Problem 3 on Sample 2.

Problem 5. Find all points on the curve $\vec{r}(t) = < 2t^2, t^3 + 3t, t^2 + 1 >$ where the curvature is zero.

$r' = \langle 4t, 3t^2+3, 2t \rangle$

$r'' = \langle 4, 6t, 2 \rangle$

$r' \times r'' = \langle 6-6t^2, 0, 12t^2-12 \rangle$

This is $\vec{0}$ if $t = \pm 1$. $|r' \times r''|$ is 0.

And this is when $\kappa = \frac{|r' \times r''|}{|r'|^3}$ is 0.

So: $(2, 4, 2)$ and $(2, -4, 2)$

Problem 6. a) Set up the integral for the length of the curve

$$\vec{r}(t) = < t, t^2, t^3 >$$

between the point $(1, 1, 1)$ and the point $(-1, 1, -1)$.

$t=1$, $t=-1$

$\vec{v} = \langle 1, 2t, 3t^2 \rangle$

$$L = \int_{-1}^{1} \sqrt{1 + 4t^2 + 9t^4}\, dt$$

b) Find the unit tangent vector at the point $(-1, 1, -1)$. $t = -1$

$$\vec{T} = \frac{\vec{v}}{|\vec{v}|} = \frac{\langle 1, -2, 3 \rangle}{\sqrt{1+4+9}} = \left\langle \frac{1}{\sqrt{14}}, \frac{-2}{\sqrt{14}}, \frac{3}{\sqrt{14}} \right\rangle$$

c) Find the normal and tangential components of the acceleration at the point $(-1, 1, -1)$.

$\vec{a} = \langle 0, 2, 6t \rangle \underset{t=-1}{=} \langle 0, 2, -6 \rangle$

Tangential: $\frac{\vec{v} \cdot \vec{a}}{|\vec{v}|} = \frac{-22}{\sqrt{14}}$

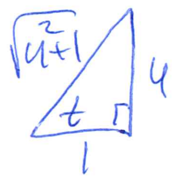
Normal: $\frac{|\vec{v} \times \vec{a}|}{|\vec{v}|} = \frac{2\sqrt{3^2+3^2+1^2}}{\sqrt{14}} = \frac{2\sqrt{19}}{\sqrt{14}}$

$$\begin{array}{r} \langle 1, -2, 3 \rangle \\ \times \quad \langle 0, 2, -6 \rangle \\ \hline \langle 6, 6, 2 \rangle \end{array}$$

Problem 7. a) Reparametrize the curve

$$(x, y, z) = (\sin t, \cos t, \tan t), \quad 0 \le t < \frac{\pi}{3}$$

using the new variable $u = \tan t$. Simplify your answer.



$\sin t = \frac{u}{\sqrt{u^2+1}}$

$\cos t = \frac{1}{\sqrt{u^2+1}}$

$\left(\frac{u}{\sqrt{u^2+1}}, \frac{1}{\sqrt{u^2+1}}; u\right) \quad 0 \le u < \sqrt{3}$

b) For the new parametrization, find the acceleration at the point $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 1)$. ← $u=1$.

$\frac{d\vec{r}}{du} = \left\langle \frac{1\cdot\sqrt{u^2+1} - \frac{u\cdot 2u}{2\sqrt{u^2+1}}}{u^2+1}, \frac{-\frac{1\cdot 2u}{2\sqrt{u^2+1}}}{u^2+1}, 1 \right\rangle = \langle -(u^2+1)^{-3/2} ..., -u\cdot(u^2+1)^{-3/2}, 1 \rangle$

$\frac{d^2\vec{r}}{du^2} = \langle \frac{3}{2}(u^2+1)^{-5/2}\cdot 2u, -(u^2+1)^{-3/2} + \frac{3}{2}u(u^2+1)^{-5/2}\cdot 2u, 0 \rangle \overset{u=1}{=} \left\langle \frac{3}{4\sqrt{2}}, \frac{1}{4\sqrt{2}}, 0 \right\rangle$

$-\frac{1}{2\sqrt{2}} + \frac{3}{4\sqrt{2}}$

Problem 8. a) Find the distance between the point $P = (1, 2, 3)$ and the plane $x + y - 2z = 8$.

$d = \left|\frac{1\cdot 1 + 1\cdot 2 - 2\cdot 3 - 8}{\sqrt{1^2+1^2+(-2)^2}}\right| = \left|\frac{-11}{\sqrt{6}}\right| = \frac{11}{\sqrt{6}}$

b) Find the parametric **and** the symmetric equations of the perpendicular from P to that plane.

Parametric: $(x, y, z) = (1+t, 2+t, 3-2t)$

Symmetric: $x - 1 = y - 2 = \frac{z-3}{-2}$

c) Find the cooridnates of the point at which the perpendicular intersects the plane.

$1(1+t) + 1\cdot(2+t) - 2(3-2t) = 8$

$6t = 11$

$t = \frac{11}{6}$

Point: $\left(\frac{17}{6}, \frac{23}{6}, -\frac{2}{3}\right)$