Department of Mathematical Sciences

Math 227 Calculus Spring 2016 Exam 2

DO NOT TURN OVER THIS PAGE UNTIL INSTRUCTED TO DO SO

NAME (Printed): ____

INSTRUCTOR: ____

SECTION NO.: ____

When instructed, turn over this cover page and begin the test. You will have 90 minutes to complete the test. If you have any questions, raise your hand and wait for the proctor to come to your seat.

This test is 5 pages long and contains 5 problems, some with several parts. Write your work on the test papers. If you need extra space, use the backs of the pages and say so on the front. **You must show all necessary work for each problem.** Solutions presented with no supporting work may receive no credit. Numerical answers should be presented as exact mathematical expressions, simplified as appropriate, not by a decimal approximation, unless explicitly required by the problem.

YOU MAY NOT USE NOTES, CELL PHONES, CALCULATORS OR LAPTOPS AT ANY TIME DURING THE TEST PERIOD. Good luck!

FOR GRADING PURPOSES, DO NOT WRITE IN THIS SPACE

Problem	Points	Credit
1	30	
2	15	
3	32	
4	30	
5	18	
Total	125	

(1) (30 Points) For each power series below find the **interval of convergence, checking endpoints**. SHOW YOUR WORK. Include the name and details of the test(s) used when checking the endpoints.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(x-3)^n}{\sqrt{n}\; 4^n}$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{5^n\; x^{2n}}{(2n)!}$

(1) (Continued) (c) $\displaystyle\sum_{n=1}^{\infty} \frac{n\,(x+2)^n}{3^n}$

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(2) (15 Points) Find the second degree Taylor polynomial $T_2(x)$ which approximates the function $f(x) = \sqrt[4]{x^3} = x^{3/4}$ at $a = 16$.

(3) (32 Points) Write a **power series representation** for each function and give its **radius of convergence**. Use summation notation to express all terms. Write out and simplify any combinatorial expressions for coefficients.

(a) $\cos(-x^3)$

(b) $\ln(1 - x^2)$

(c) $\dfrac{x^3}{5 + x}$

(d) $(1 - x)^{-1/4}$

(4) (30 Points) Use your knowledge of how functions are represented by power series to determine the exact values for each of the following series. Write your answers in terms of a known function and explain how you used a power series to get the answer.

(a) $\displaystyle\sum_{n=0}^{\infty} \binom{\frac{-1}{2}}{n} \frac{1}{3^n}$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n\ 5^n}{(2n)!}$

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{3^{2n+1}(2n+1)!}$

(5) (18 Points, 6 Pts each) Simplify your answer in each part.

(a) Write $\dfrac{d}{dx}\left[\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \; x^{2n+1}}{(2n+1)!}\right]$ as a **power series** and as a **function**.

(b) Write $\displaystyle\int_0^1 \left[\sum_{n=0}^{\infty} \frac{x^n}{n!}\right] dx$ as a **series** and give its **value**.

(c) The **power series** $\displaystyle\sum_{n=2}^{\infty} \binom{k}{n} n(n-1)x^{n-2}$ converges to what **function** for $|x| < 1$?

Math 227 Calculus 2 Spring 2016 Exam 2 Solutions

(1) (30 Points) Find the **interval of convergence**, checking endpoints.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(x-3)^n}{\sqrt{n}\ 4^n}$

(10 Points) The Ratio Test gives $\left|\frac{(x-3)^{n+1}}{\sqrt{n+1}\ 4^{n+1}} \frac{\sqrt{n}\ 4^n}{(x-3)^n}\right| = \sqrt{\frac{n}{n+1}}\frac{|x-3|}{4}$. As $n \to \infty$ this ratio goes to $L = \frac{|x-3|}{4}$ so that $L < 1$ iff $|x-3| < 4$ iff $-1 < x < 7$. We must check for convergence at the endpoints separately. At $x = -1$ the series is $\sum_{n=1}^{\infty} \frac{(-4)^n}{\sqrt{n}\ 4^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ which converges by the AST since $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$ and $\frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}}$. At $x = 7$ the series is $\sum_{n=1}^{\infty} \frac{4^n}{\sqrt{n}\ 4^n} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ which is a diverging p-series with $p = \frac{1}{2} \leq 1$. So the complete domain of convergence is $[-1, 7)$, that is, $-1 \leq x < 7$.

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{5^n\ x^{2n}}{(2n)!}$

(10 Points) The Ratio Test gives $\left|\frac{5^{n+1}\ x^{2n+2}}{(2n+2)!} \frac{(2n)!}{5^n\ x^{2n}}\right| = \frac{5|x^2|}{(2n+2)(2n+1)}$. As $n \to \infty$ for any fixed x, this ratio goes to zero, so the series converges (absolutely) for all real x, that is, on $(-\infty, \infty)$. There are no endpoints to check.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{n\ (x+2)^n}{3^n}$

(10 Points) The Ratio Test gives $\left|\frac{(n+1)(x+2)^{n+1}}{3^{n+1}} \frac{3^n}{n(x+2)^n}\right| = \frac{n+1}{n}\frac{|x+2|}{3}$. As n goes to infinity, this ratio goes to $L = \frac{|x+2|}{3}$ so that $L < 1$ iff $|x+2| < 3$ iff $-3 < x+2 < 3$ iff $-5 < x < 1$. We must check for convergence at the endpoints separately. At $x = 1$ the series is $\displaystyle\sum_{n=1}^{\infty} \frac{n\ (3)^n}{3^n} = \sum_{n=1}^{\infty} n$ which diverges by TFD since $\lim_{n\to\infty} n = \infty$. At $x = -5$ the series is $\displaystyle\sum_{n=1}^{\infty} \frac{n\ (-3)^n}{3^n} = \sum_{n=1}^{\infty} n(-1)^n$ which diverges by TFD since $\lim_{n\to\infty} n(-1)^n$ DNE. So the interval of convergence is $(-5, 1)$, that is, $-5 < x < 1$.

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(2) (15 Points) Find the second degree Taylor polynomial $T_2(x)$ which approximates the function $f(x) = \sqrt[4]{x^3} = x^{3/4}$ at $a = 16$. We have
$f^{(1)}(x) = \frac{3}{4}x^{-1/4}$ and $f^{(2)}(x) = \frac{-3}{16}x^{-5/4}$. Then $f(16) = (16)^{3/4} = 8$,
$f^{(1)}(16) = \frac{3}{4}(16)^{-1/4} = \frac{3}{8}$ and $f^{(2)}(16) = \frac{-3}{16}(16)^{-5/4} = \frac{-3}{(16)(32)} = \frac{-3}{512}$ so
$T_2(x) = f(16) + \dfrac{f^{(1)}(16)}{1!}(x-16) + \dfrac{f^{(2)}(16)}{2!}(x-16)^2 = 8 + \dfrac{3}{8}(x-16) - \dfrac{3}{1024}(x-16)^2.$

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(3) (32 Points) Write a power series representation for each function and give its radius of convergence. Use summation notation to express all terms. Write out and simplify any combinatorial expressions for coefficients.

(a) (8 Points) $\cos(-x^3) = \displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{(-x^3)^{2n}}{(2n)!} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{6n}}{(2n)!}$ with radius $R = \infty$ by substituting $t = -x^3$ in $\cos(t) = \displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{t^{2n}}{(2n)!}$, which has radius $R = \infty$.

(b) (8 Points) $\ln(1-x^2) = \displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(-x^2)^n}{n} = \sum_{n=1}^{\infty}\frac{(-1)x^{2n}}{n}$ with radius $R = 1$ by substituting $t = -x^2$ in $\ln(1+t) = \displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}t^n}{n}$ which has radius $R = 1$.

(c) (8 Points) $\dfrac{x^3}{5+x} = \dfrac{x^3}{5}\dfrac{1}{(1+\frac{x}{5})} = \dfrac{x^3}{5}\displaystyle\sum_{n=0}^{\infty}\left(\frac{-x}{5}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n x^{n+3}}{5^{n+1}}$
with radius $R = 5$ because $\displaystyle\sum_{n=0}^{\infty}\left(\frac{-x}{3}\right)^n$ is a geometric series that converges for $|\frac{-x}{5}| < 1$ so $|x| < 5$ gives radius 5.

(d) (8 Points) $(1-x)^{-1/4}$ is a binomial series $(1+t)^k = \displaystyle\sum_{n=0}^{\infty}\binom{k}{n}t^n$ with $t = -x$ and $k = \frac{-1}{4}$, with radius $R = 1$. It's power series representation is:

$$\sum_{n=0}^{\infty}\binom{\frac{-1}{4}}{n}(-1)^n x^n = 1 + \sum_{n=1}^{\infty}\frac{(\frac{-1}{4})(\frac{-5}{4})\cdots(\frac{-1}{4}-n+1)}{n!}(-1)^n x^n$$

$$= 1 + \sum_{n=1}^{\infty}\frac{(1)(5)(9)\cdots(4n-3)}{n!\ 4^n}x^n.$$

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(4) (30 Points) Use your knowledge of how functions are represented by power series to determine the exact values of the following infinite series. Write your answers in terms of a known function and explain how you used a power series to get the answer.

(a) $\displaystyle\sum_{n=0}^{\infty}\binom{\frac{-1}{2}}{n}\frac{1}{3^n} = \left(1+\frac{1}{3}\right)^{-1/2} = \left(\frac{4}{3}\right)^{-1/2} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$ since the binomial series $\displaystyle(1+x)^k = \sum_{n=0}^{\infty}\binom{k}{n}x^n$ has radius $R=1$. Use $k=\frac{-1}{2}$ and $x=\frac{1}{3}$.

(b) $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n\ 5^n}{(2n)!} = \sum_{n=0}^{\infty}\frac{(-1)^n\ \sqrt{5}^{2n}}{(2n)!} = \cos(\sqrt{5})$ since $\displaystyle\cos(x) = \sum_{n=0}^{\infty}\frac{(-1)^n\ x^{2n}}{(2n)!}$ where $R=\infty$. Used $x=\sqrt{5}$.

(c) $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n\pi^{2n+1}}{3^{2n+1}(2n+1)!} = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$ since $\displaystyle\sin(x) = \sum_{n=0}^{\infty}\frac{(-1)^n\ x^{2n+1}}{(2n+1)!}$ where $R=\infty$. Used $x=\frac{\pi}{3}$.

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(5) (18 Points, 6 Pts each) Simplify your answer in each part.

(a) $\displaystyle\frac{d}{dx}\left[\sum_{n=0}^{\infty}\frac{(-1)^n\ x^{2n+1}}{(2n+1)!}\right] = \sum_{n=0}^{\infty}\frac{(-1)^n\ (2n+1)x^{2n}}{(2n+1)!} = \sum_{n=0}^{\infty}\frac{(-1)^n\ x^{2n}}{(2n)!} = \cos(x)$ as a **power series** and as a **function**. The left side is the derivative of $\sin(x)$.

(b) $\displaystyle\int_0^1\left[\sum_{n=0}^{\infty}\frac{x^n}{n!}\right]dx = \sum_{n=0}^{\infty}\frac{1}{n!}\frac{x^{n+1}}{n+1}\Bigg|_{x=0}^{x=1} = \sum_{n=0}^{\infty}\frac{1}{n!}\frac{1}{n+1} = \sum_{n=0}^{\infty}\frac{1}{(n+1)!} = e-1$ since $\displaystyle e^1 = \sum_{n=0}^{\infty}\frac{1}{n!}$. Also, $\displaystyle\int_0^1\left[\sum_{n=0}^{\infty}\frac{x^n}{n!}\right]dx = \int_0^1 e^x\,dx = e^1 - e^0 = e-1.$

(c) The **power series** $\displaystyle\sum_{n=2}^{\infty}\binom{k}{n}n(n-1)x^{n-2}$ converges to what **function** for $|x|<1$?

Taking two derivatives of the binomial series (which converges for $|x|<1$) gives $\displaystyle k(k-1)(1+x)^{k-2} = \frac{d^2}{dx^2}(1+x)^k = \frac{d^2}{dx^2}\sum_{n=0}^{\infty}\binom{k}{n}x^n = \sum_{n=0}^{\infty}\binom{k}{n}n(n-1)x^{n-2}.$

So that series converges to the function $k(k-1)(1+x)^{k-2}$ for $|x|<1$.
