Math 227 (White) **Midterm Exam**
Infinite Series **November 17, 2017**

Do NOT turn over this page until instructed to begin.

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Instructions

Write clear, careful, neat solutions to the questions in the space provided.

No books, no notes, no electronic devices (calculators, cell phones, smart watches, etc.) allowed!

Write all your work on the test–nothing else will be graded. **You must show all your work and justify your answers.** Solutions presented without sufficient supporting work may receive no credit. Your work must be legible, and numerical answers should be presented as exact mathematical expressions, simplified as appropriate, and not by a decimal approximation, unless explicitly required by the problem.

On some problems you may be asked to use a specific method to solve the problem (for instance, "Use the Fundamental Theorem of Calculus to find..."). On all other problems, you may use any method we have covered. You may not use methods that we have not covered.

Wandering Eyes Policy

You must keep your eyes on your own work at all times. If you are found looking around, you will be warned once, and only once. A second infraction may result in automatic zero on this test, and possibly a referral to the Harpur College Academic Honesty Committee.

Duration of the Test

This is a timed test designed for one class period. You will start the test when your proctor tells you to start, and you MUST stop working when your proctor tells you to stop.

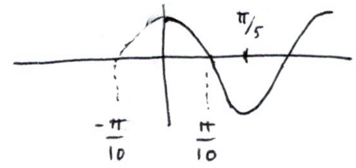
Some Useful Identities

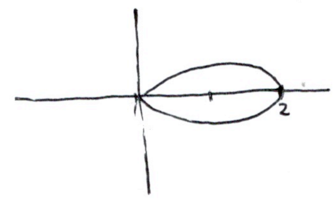
$$\sin^2(x) = \frac{1}{2}(1 - \cos(2x)) \qquad \cos^2(x) = \frac{1}{2}(1 + \cos(2x))$$

1	2	3	4	5	6	Total
15 pts	20 pts	25 pts	5 pts	20 pts	15 pts	100 pts

1. (15 points) Answer these questions about the polar curve $r = 2\cos(5\theta)$, for $0 \leq \theta \leq 2\pi$.

(a) Compute the area enclosed by *one loop* of the graph of the curve.
[Hint: A decent sketch of the graph may help.]





$$\frac{1}{2}\int_{-\pi/10}^{\pi/10} (2\cos(5\theta))^2 \, d\theta$$

$$= 2\int_{-\pi/10}^{\pi/10} \cos^2(5\theta)\, d\theta$$

$$= 2\int_{-\pi/10}^{\pi/10} \frac{1}{2}(1+\cos(10\theta))\, d\theta$$

$$= \int_{-\pi/10}^{\pi/10} 1 + \cos(10\theta)\, d\theta \qquad \left[\text{Alt. limits: } \frac{\pi}{10} \text{ to } \frac{3\pi}{10}\right]$$

$$= \theta + \frac{1}{10}\sin(10\theta)\Big]_{-\pi/10}^{\pi/10}$$

$$= \left(\frac{\pi}{10} + \frac{1}{10}\sin(\pi)\right) - \left(-\frac{\pi}{10} + \frac{1}{10}\sin(-\pi)\right) = \frac{\pi}{5}$$

(b) Set up the integral for the arc length of one loop of the curve.
Do NOT evaluate the integral.

$$r(\theta) = 2\cos(5\theta)$$
$$r'(\theta) = -10\sin(5\theta)$$

$$\int_{-\pi/10}^{\pi/10} \sqrt{(2\cos(5\theta))^2 + (-10\sin(5\theta))^2}\, d\theta$$

$$= \int_{-\pi/10}^{\pi/10} \sqrt{4\cos^2(5\theta) + 100\sin^2(5\theta)}\, d\theta$$

2. (20 points) Determine whether each sequence $\{a_n\}_{n=1}^{\infty}$ converges or diverges. If possible, find its limit. Show all necessary work to justify your answer.

(a) $a_n = \dfrac{(\ln(n))^2}{n}$

Let $f(x) = \dfrac{(\ln(x))^2}{x}$ and observe $a_n = f(n)$.

Then $\lim\limits_{n\to\infty} a_n = \lim\limits_{x\to\infty} f(x)$.

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{2\ln(x)\cdot\frac{1}{x}}{1} \quad \text{(L'Hôpital's Theorem)}$$

$$= \lim_{x\to\infty} \frac{2\ln(x)}{x}$$

$$= \lim_{x\to\infty} \frac{2}{x} \quad \text{(L'Hôpital's Theorem)}$$

$$= 0.$$

Thus $\lim\limits_{n\to\infty} a_n = 0$ as well.

(b) $a_n = \dfrac{5\sqrt{n}}{\sqrt{n}+3}$

$$\lim_{n\to\infty} \frac{5\sqrt{n}}{\sqrt{n}+3} = \lim_{n\to\infty} \frac{\left(\frac{1}{\sqrt{n}}\right)5\sqrt{n}}{\left(\frac{1}{\sqrt{n}}\right)(\sqrt{n}+3)}$$

$$= \lim_{n\to\infty} \frac{5}{1+(3/\sqrt{n})}$$

$$= 5 \quad \left(\text{using } \lim_{n\to\infty} \frac{3}{\sqrt{n}} = 0 \text{ and the "limit laws"}\right).$$

(c) $a_n = \left(1 - \dfrac{1}{3n}\right)^n$

Let $f(x) = \left(1 - \frac{1}{3x}\right)^x$ and note $a_n = f(n)$, so $\lim\limits_{n\to\infty} a_n = \lim\limits_{x\to\infty} f(x)$.

$$\log \lim_{x\to\infty} \left(1-\tfrac{1}{3x}\right)^x = \lim_{x\to\infty} \log\left(1-\tfrac{1}{3x}\right)^x \quad \text{(because } \log(x) \text{ is continuous)}$$
$$= \lim_{x\to\infty} x \log\left(1-\tfrac{1}{3x}\right)$$
$$= \lim_{x\to\infty} \frac{\log\left(1-\frac{1}{3x}\right)}{\left(\frac{1}{x}\right)}$$
$$= \lim_{x\to\infty} \frac{\left(\frac{1}{1-(\frac{1}{3x})}\right)\cdot\left(\frac{1}{3x^2}\right)}{\left(\frac{-1}{x^2}\right)}$$
$$= \lim_{x\to\infty} \frac{-1}{3 - \frac{1}{x}}$$
$$= -\frac{1}{3}$$

Thus
$$\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x) = e^{-\frac{1}{3}} = \frac{1}{\sqrt[3]{e}}.$$

(d) $a_n = \dfrac{3^n}{n!}$ [Hint: Use the Squeeze Theorem].

$$a_1 = \frac{3}{1} < \frac{9}{2}\cdot\frac{3}{1}$$
$$a_2 = \frac{3}{1}\cdot\frac{3}{2} < \frac{9}{2}\cdot\frac{3}{2}$$
$$a_3 = \frac{3}{1}\cdot\frac{3}{2}\cdot\frac{3}{3} < \frac{9}{2}\cdot\frac{3}{3}$$
$$a_4 = \frac{3}{1}\cdot\frac{3}{2}\cdot\frac{3}{3}\cdot\frac{3}{4} < \frac{9}{3}\cdot\frac{3}{4}$$
$$a_5 = \underbrace{\frac{3}{1}\cdot\frac{3}{2}\cdot\frac{3}{3}}\cdot\frac{3}{4}\cdot\frac{3}{5} < \frac{9}{2}\cdot\frac{3}{5}$$

$$a_n = \frac{3\cdot 3\cdot 3\cdot 3\cdots 3}{1\ 2\ 3\ 4 \cdots n} < \frac{9}{2}\cdot\frac{3}{n}$$

$$0 < a_n < \frac{9}{2}\cdot\frac{3}{n}$$

Since $\lim\limits_{n\to\infty} 0 = 0$ and $\lim\limits_{n\to\infty} \frac{9}{2}\cdot\frac{3}{n} = 0$,

by the squeeze theorem, $\lim\limits_{n\to\infty} a_n = 0$ as well.

3. (25 points) In each part test the given series for convergence or divergence. Be sure to indicate which test you use and *all* of the steps required to apply the test. If possible, compute the sum (limit) of the series. Indicate your conclusion clearly.

(a) $\sum_{n=2}^{\infty} \ln\left(\frac{3n^3-5}{n^3+8}\right)$

$$\lim_{n\to\infty} \ln\left(\frac{3n^3-5}{n^3+8}\right) = \ln\left(\lim_{n\to\infty} \frac{3n^3-5}{n^3+8}\right)$$ (because $\ln(x)$ is continuous)

$$= \ln(3) \neq 0$$

By the divergence test, series (a) diverges.

(b) $\sum_{n=1}^{\infty} \frac{2^{n+2}}{(-5)^n}$ (*)

$$(*) = \sum_{n=1}^{\infty} (-1)^n \cdot 4 \cdot \left(\frac{2}{5}\right)^n$$

$$= \sum_{n=1}^{\infty} 4\left(\frac{-2}{5}\right)^n$$ (is geometric)

$$= \frac{a}{1-r}$$ $a = \frac{-8}{5}$ $r = \left(\frac{-2}{5}\right)$ $|r| = \frac{2}{5} < 1$

$$= \frac{-8}{5}\left(\frac{1}{1+\frac{2}{5}}\right)$$

$$= \frac{-8}{5} \cdot \frac{5}{7} = \frac{-8}{7}$$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{4\sqrt{n}-1}{n^2+2\sqrt{n}}$

Comparison test: $\quad 0 \leq \dfrac{4\sqrt{n}-1}{n^2+2\sqrt{n}} \leq \dfrac{4\sqrt{n}}{n^2} = \dfrac{4}{n^{3/2}}$

$\displaystyle\sum_{n=1}^{\infty} \frac{4}{n^{3/2}}$ is a convergent p-series $(p = 3/2 > 1)$

so by the comparison test series (c) converges.

Alt: Use limit comparison

$$\lim_{n\to\infty} \frac{\left(\frac{4\sqrt{n}-1}{n^2+2\sqrt{n}}\right)}{\left(\frac{1}{n^{3/2}}\right)} = \lim_{n\to\infty} \frac{4n^2-n^{3/2}}{n^2+2\sqrt{n}}$$

$$= \lim_{n\to\infty} \frac{4-\left(\frac{1}{\sqrt{n}}\right)}{1+\left(\frac{2}{n^{3/2}}\right)} = 4 > 0.$$

Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ is a convergent p-series $(p = \frac{3}{2} > 1)$ and $4 > 0$, the series (c) also converges.

(d) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{3n+2}\right)^n$

$$a_n = \left(\frac{n}{3n+2}\right)^n$$

Use root test:

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{n}{3n+2} = \frac{1}{3}.$$

Since $\frac{1}{3} < 1$, the series (d) converges by the root test.

4. (5 points) Give a clear statement of the **Alternating Series Test** for convergence of a series $\sum_{n=1}^{\infty} a_n$. (Do not forget to state *all* the required *hypotheses*.)

If $\sum_{n=1}^{\infty} a_n$ is an alternating series and

① $|a_{n+1}| < |a_n|$ for all n

and ② $\lim_{n\to\infty} |a_n| = 0$

then $\sum_{n=1}^{\infty} a_n$ converges

Alt: $b_n = |a_n|$

$b_{n+1} < b_n$ for all n

$\lim_{n\to\infty} b_n = 0$

5. (20 points) Test the following series for *absolute convergence*, *conditional convergence*, or *divergence*. Explain what tests you are applying and how you apply them. Be sure to make your final conclusion clear.

(a) $\sum_{n=1}^{\infty} \frac{n^2 2^{n+1}}{3^n}$ $\qquad a_n = \frac{n^2 2^{n+1}}{3^n}$

Apply the ratio test:

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{(n+1)^2\, 2^{(n+1)+1}}{3^{n+1}} \cdot \frac{3^n}{n^2 \cdot 2^{n+1}}$$

$$= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^2 \cdot \frac{2}{3}$$

$$= \frac{2}{3} \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^2$$

$$= \frac{2}{3} < 1$$

Since $\frac{2}{3} < 1$, we conclude that (a) converges (absolutely) by the ratio test.

(b) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n+3}$ $\qquad a_n = \frac{(-1)^{n+1}}{2n+3} \qquad b_n = |a_n| = \frac{1}{2n+3}$

Since $\frac{1}{2n+3} > 0$, this series is alternating.

Clearly, $\frac{1}{2(n+1)+3} < \frac{1}{2n+3}$ for all n, since

$$2(n+1)+3 > 2n+3.$$

Also $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{2n+3} = 0.$

By the alternating series test, series (b) converges.

Use the limit comparison test to compare $\sum_{n=1}^{\infty} |a_n|$ to the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$.

$$\lim_{n\to\infty} \frac{\left(\frac{1}{2n+3}\right)}{\left(\frac{1}{n}\right)} = \lim_{n\to\infty} \frac{n}{2n+3} = \frac{1}{2} > 0$$

(or... $\lim_{n\to\infty} \frac{\left(\frac{1}{n}\right)}{\left(\frac{1}{2n+3}\right)} = \lim_{n\to\infty} \frac{2n+3}{n} = \lim_{n\to\infty} 2+\frac{3}{n} = 2 > 0$).

Since the harmonic series diverges, $\sum_{n=1}^{\infty} |a_n|$ also diverges.

Series (b) converges conditionally.

6. (15 points) Consider the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n^3 - 2}$

(a) Show that the alternating series above converges.

Set $a_n = \frac{(-1)^{n+1}}{2n^3-2}$ and $b_n = |a_n| = \frac{1}{2n^3-2}$.

Then $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^n b_n$ is alternating.

Clearly $b_{n+1} < b_n$ because $2(n+1)^3-2 > 2n^3-2$,

which tells us $\frac{1}{2(n+1)^3-2} < \frac{1}{2n^3-2}$ for all n.

Thus $\{b_n\}_{n=1}^{\infty}$ is decreasing.

Also $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{2n^3-2} = 0$.

By the alternating series test $\sum_{n=1}^{\infty} a_n$ converges.

(b) Use the alternating series estimation theorem to find the least integer n for which the n-th partial sum, S_n, is within an error of $1/1000$. In other words, find the smallest n such that you can be sure that $|S - S_n| < 1/1000$.

We know that $|S-S_n| < b_{n+1}$ so we need to find the first integer n such that

$$\frac{1}{2(n+1)^3-2} = b_{n+1} < \frac{1}{1000}$$

$$2(n+1)^3-2 > 1000$$

$$2(n+1)^3 > 1002$$

$$(n+1)^3 > 501$$

Note: $7^3 = 343$
$8^3 = 2^9 = 512$

Take $n+1 = 8$, so $\boxed{n = 7.}$