Department of Mathematical Sciences

Math 227 Calculus Spring 2016 Exam 1V1

DO NOT TURN OVER THIS PAGE UNTIL INSTRUCTED TO DO SO

NAME (Printed): ______________________________

INSTRUCTOR: ______________________________

SECTION NO.: ______________________________

When instructed, turn over this cover page and begin the test. You will have 90 minutes to complete the test. If you have any questions, raise your hand and wait for the proctor to come to your seat.

This test is 6 pages long and contains 7 problems, some with several parts. Write your work on the test papers. If you need extra space, use the backs of the pages and say so on the front. You must show all necessary work for each problem. Solutions presented with no supporting work may receive no credit. Numerical answers should be presented as exact mathematical expressions, simplified as appropriate, not by a decimal approximation, unless explicitly required by the problem.

YOU MAY NOT USE NOTES, CELL PHONES, CALCULATORS OR LAPTOPS AT ANY TIME DURING THE TEST PERIOD. Good luck!

Here are some useful identities:

$$\sin^2(x) = \frac{1-\cos(2x)}{2} \qquad \cos^2(x) = \frac{1+\cos(2x)}{2}$$

FOR GRADING PURPOSES, DO NOT WRITE IN THIS SPACE

Problem	Points	Credit
1	25	
2	10	
3	20	
4	25	
5	20	
6	20	
7	5	
Total	125	

(1) (25 Points, 5 Points each) Determine whether each sequence $\{a_n\}_{n=1}^{\infty}$ converges or diverges, and if it converges, then find its limit.

(a) $a_n = \sqrt{\dfrac{n}{n+1} + \dfrac{n}{n-1}}$ for $n \geq 2$

(b) $a_n = \dfrac{4n^3 - 3n^2 + n - 1}{5n^3 + 4n^2 + 3n + 2}$

(c) $a_n = \dfrac{\sqrt[3]{n}}{\ln(n)}$ for $n \geq 2$

(1) (Continued)

(d) $a_n = \arcsin\left(\dfrac{1+n\sqrt{3}}{1+2n}\right)$

(e) $a_n = \dfrac{(1)(3)(5)\cdots(2n-1)}{(2)(4)(6)\cdots(2n)(2n+2)}$ (Hint: Use the Squeeze Theorem.)

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(2) (10 Points) A **convergent** sequence of **positive** numbers satisfies the recursive relation $a_n a_{n+1} = a_n + 1$. Find $\lim_{n\to\infty} a_n$.

(3) (20 Points) In each part test the series for convergence or divergence. Write all steps of the test you use.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^5 + 2n^3 + 1}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[4]{n^9 - n^5 + 2}}$

(4) (25 Points) Test the following series for absolute convergence, conditional convergence, or divergence. Explain what tests you are applying and how you apply them.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n^2+1}}$

(b) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{3n}{2n+1}\right)^{2n}$

(5) (20 Points) Answer these questions about the polar curve $r = \sqrt{\theta}$ for $0 \leq \theta \leq 2\pi$.

(a) Find the slope of the tangent line to the curve at $\theta = \frac{\pi}{3}$.

(b) Find the area of the polar region determined by the curve.

(c) Set up the integral for the arclength of that polar curve, but DO NOT EVALUATE IT.

(6) (20 Points) The alternating series $\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt[5]{k}}$ converges to S and has N^{th} partial sum $S_N = \sum_{k=1}^{N} \frac{(-1)^k}{\sqrt[5]{k}}$. Find the smallest N such that you can be sure $|S - S_N| < \frac{1}{1000}$.

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(7) (5 Points) State the Monotone Convergence Theorem exactly.

Math 227 Calculus Spring 2016 Exam 1V1 Solutions

(1) (25 Points, 5 Points Each) Determine whether each sequence $\{a_n\}_{n=1}^{\infty}$ converges or diverges, and if it converges, then find its limit.

(a) $a_n = \sqrt{\dfrac{n}{n+1} + \dfrac{n}{n-1}} = \sqrt{\dfrac{2n^2}{n^2-1}} = \sqrt{\dfrac{2}{1-\frac{1}{n^2}}} \to \sqrt{2}$ as $n \to \infty$ converges since $\dfrac{1}{n^2} \to 0$.

(b) $a_n = \dfrac{4n^3 - 3n^2 + n - 1}{5n^3 + 4n^2 + 3n + 2} = \dfrac{4 - \frac{3}{n} + \frac{1}{n^2} - \frac{1}{n^3}}{5 + \frac{4}{n} + \frac{3}{n^2} + \frac{2}{n^3}} \to \dfrac{4}{5}$ converges as $n \to \infty$.

(c) $a_n = \dfrac{\sqrt[3]{n}}{\ln(n)} = f(n)$ where $f(x) = \dfrac{\sqrt[3]{x}}{\ln(x)}$. Using L'Hospital's Rule ($\frac{\infty}{\infty}$-type):
$\displaystyle\lim_{x\to\infty} \frac{\sqrt[3]{x}}{\ln(x)} = \lim_{x\to\infty} \frac{\frac{1}{3}x^{-2/3}}{x^{-1}} = \lim_{x\to\infty} \frac{1}{3}x^{1/3} \to \infty$ diverges.

(d) $a_n = \arcsin\left(\dfrac{1+n\sqrt{3}}{1+2n}\right) \to \arcsin(\sqrt{3}/2) = \dfrac{\pi}{3}$ converges since $\dfrac{1+n\sqrt{3}}{1+2n} \to \dfrac{\sqrt{3}}{2}$ as $n \to \infty$, and $\arcsin(x)$ is continuous at $\sqrt{3}/2$.

(e) $a_n = \dfrac{(1)(3)(5)\cdots(2n-1)}{(2)(4)(6)\cdots(2n)(2n+2)}$ (Hint: Use the Squeeze Theorem.)

Since $2i - 1 < 2i$ for $1 \le i \le n$, we get

$$0 \le a_n = \frac{1}{2}\frac{3}{4}\frac{5}{6}\cdots\frac{2n-1}{2n}\frac{1}{2n+2} < \frac{1}{2n+2}$$

This means that $0 \le a_n \le \frac{1}{2n+2}$, and since $\displaystyle\lim_{n\to\infty} \frac{1}{2n+2} = 0$, the Squeeze Theorem says the sequence converges to 0.

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(7) (10 Points) A **convergent** sequence of **positive** numbers satisfies the recursive relation $a_n a_{n+1} = a_n + 1$. Find $\lim_{n\to\infty} a_n$.

Taking the limit as $n \to \infty$ of the recursive relation, and using that $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} a_{n+1}$, get

$$(L)(L) = L + 1 \quad \text{so} \quad 0 = L^2 - L - 1 \quad \text{so } L = \frac{1+\sqrt{5}}{2} \quad \text{is the only positive solution.}$$

(3) (20 Points) In each part test the series for convergence or divergence. Write all steps of the test you use.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^5 + 2n^3 + 1}$

(10 Points) Since $0 \leq \frac{1}{n^5+2n^3+1} \leq \frac{1}{n^5}$ for all $n \geq 1$, and $\sum_{n=1}^{\infty} \frac{1}{n^5}$ is a convergent p-series with $p = 5 > 1$, the series converges by the Comparison test.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[4]{n^9 - n^5 + 2}}$

(10 Points) Apply the Limit Comparison test where the series $\sum b_n$ being compared to is the convergent p-series $\sum \frac{1}{n^{9/4}}$ with $p = 9/4 > 1$. We have

$$\lim_{n\to\infty} \frac{\frac{1}{\sqrt[4]{n^9-n^5+2}}}{\frac{1}{\sqrt[4]{n^9}}} = \lim_{n\to\infty} \sqrt[4]{\frac{n^9}{n^9 - n^5 + 2}} = \lim_{n\to\infty} \sqrt[4]{\frac{1}{(1 - \frac{1}{n^4} + \frac{2}{n^9})}} = 1 > 0$$

so the Limit Comparison test says both series have the same behavior, they both converge.

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(4) (25 Points) Test the following series for absolute convergence, conditional convergence, or divergence. Explain what tests you are applying and how you apply them.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n^2 + 1}}$

(15 Points) The series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n^2+1}}$ is an alternating series, and for $n \geq 1$ we have $\sqrt[3]{(n+1)^2 + 1} > \sqrt[3]{n^2+1}$ so $\frac{1}{\sqrt[3]{(n+1)^2+1}} < \frac{1}{\sqrt[3]{n^2+1}}$. Also, $lim_{n\to\infty} \frac{1}{\sqrt[3]{n^2+1}} = 0$, so this series converges by the Alternating Series test. Do the Limit Comparison Test of the absolute values of the terms of this series with the terms of the divergent p-series with $p = \frac{2}{3} < 1$,

$$\lim_{n\to\infty} \frac{\frac{1}{\sqrt[3]{n^2+1}}}{\frac{1}{\sqrt[3]{n^2}}} = \lim_{n\to\infty} \sqrt[3]{\frac{n^2}{n^2+1}} = \lim_{n\to\infty} \sqrt[3]{\frac{1}{(1 + \frac{1}{n^2})}} = 1 > 0$$

so the Limit Comparison test says both series have the same behavior, they both diverge. So the given series converges conditionaly.

(b) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{3n}{2n+1}\right)^{2n}$

(10 Points) Applying the root test to the series gives $L = \lim_{n\to\infty} \left(\frac{3n}{2n+1}\right)^2 = \frac{9}{4} > 1$ so the series diverges by the Root Test.

(5) (20 Points) Answer these questions about the polar curve $r = \sqrt{\theta}$ for $0 \le \theta \le 2\pi$.

(a) (10 points) Find the slope of the tangent line to the curve at $\theta = \frac{\pi}{3}$.

The slope of the tangent line to the curve for any θ is $\dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta}$.
Since $y = r\sin(\theta) = \sqrt{\theta}\sin(\theta)$ and $x = r\cos(\theta) = \sqrt{\theta}\cos(\theta)$, we have
$\dfrac{dy}{dx} = \dfrac{\frac{\sin(\theta)}{2\sqrt{\theta}} + \sqrt{\theta}\cos(\theta)}{\frac{\cos(\theta)}{2\sqrt{\theta}} - \sqrt{\theta}\sin(\theta)}$ so when $\theta = \dfrac{\pi}{3}$ this slope equals $\dfrac{\frac{\sqrt{3}/2}{2\sqrt{\pi/3}} + \sqrt{\pi/3}\frac{1}{2}}{\frac{\frac{1}{2}}{2\sqrt{\pi/3}} - \sqrt{\pi/3}\frac{\sqrt{3}}{2}} = \dfrac{3\sqrt{3}+2\pi}{3-2\sqrt{3}\pi}$.

(b) (5 points) Find the area of the polar region determined by the curve.

The area is $\displaystyle\int_0^{2\pi} \frac{1}{2}r^2 d\theta = \frac{1}{2}\int_0^{2\pi} \theta d\theta = \frac{\theta^2}{4}\Big|_{\theta=0}^{\theta=2\pi} = \frac{(2\pi)^2}{4} = \pi^2$.

(c) (5 points) Set up the integral for the arclength of that polar curve, but DO NOT EVALUATE IT.

The arclength integral is

$$\int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_0^{2\pi} \sqrt{\theta + \left(\frac{1}{2\sqrt{\theta}}\right)^2}\, d\theta = \int_0^{2\pi} \sqrt{\theta + \frac{1}{4\theta}}\, d\theta.$$

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(6) (20 Points) The alternating series $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt[5]{k}}$ converges to S and has N^{th} partial sum $S_N = \displaystyle\sum_{k=1}^{N} \frac{(-1)^k}{\sqrt[5]{k}}$. Find the smallest N such that you can be sure $|S - S_N| < \frac{1}{1000}$.

We have $|S - S_N| \le \dfrac{1}{\sqrt[5]{N+1}}$ from the Alternating Series Estimation theorem, so to guarantee the accuracy given, we need $\dfrac{1}{\sqrt[5]{N+1}} < \dfrac{1}{1000}$ which means
$10^3 = 1000 < \sqrt[5]{N+1}$ which means
$10^{15} < N+1$. The smallest N such that this is true is $N = 10^{15} = 1,000,000,000,000,000$.

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(7) (5 Points) State the Monotone Convergence Theorem exactly.

Any bounded monotone sequence converges.
