1. (10 points) Find all critical numbers of the function

\[ f(x) = \frac{x^2}{3-x} \]

\[ f'(x) = \frac{(3-x)2x - x^2(-1)}{(3-x)^2} = \frac{6x - 2x^2 + x^2}{(3-x)^2} = \frac{x(6-x)}{(3-x)^2} \]

\( f' = 0 \) when \( x = 0, \ x = 6 \)

\( f' \) DNE when \( x = 3 \) (but 3 is not in \( \text{Dom}(f) \)).

Thus critical numbers are \( x = 0 \) and \( x = 6 \).

2. (10 points) Find the limits.

a) \( \lim_{x \to -\infty} \frac{5x^3 - 3x + 1}{2x^2 + x^2} \)

\[ = \lim_{x \to -\infty} \frac{\frac{5x^3}{x^3} - \frac{3x}{x^3} + \frac{1}{x^3}}{\frac{2x^3}{x^3} + \frac{x^2}{x^3}} = \lim_{x \to -\infty} \frac{5 - \frac{3}{x} + \frac{1}{x^3}}{2 + \frac{1}{x}} \]

\[ = \frac{5}{2} \]

b) \( \lim_{x \to -\infty} \frac{\sqrt{3x^2 - 1}}{2x - 3} \)

\[ = \lim_{x \to -\infty} \frac{\sqrt{3x^2} \cdot \frac{1}{x}}{2x - 3} = \lim_{x \to -\infty} \frac{-\sqrt{x^2} \cdot \frac{1}{x}}{2 - \frac{3}{x}} = \lim_{x \to -\infty} \frac{-\sqrt{3} \cdot \frac{1}{x}}{2 - \frac{3}{x}} \]

Note that

\( \sqrt{x^2} = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases} \)

and since \( x \to -\infty \),

\( \sqrt{x^2} = -x \)

\( \Rightarrow -\sqrt{x^2} = x \)
3. (10 points) Given the derivative \( f'(x) \) of the function \( f(x) \), list all intervals on which \( f(x) \) is increasing.

\[
f'(x) = \frac{(2x - 1)^2(x - 4)}{x^3(x^2 + 3)}
\]

\( f' \) can change signs at roots of numerator or denominator - in this case:

\( x = \frac{1}{2}, \ 4, \ 0 \)

\( (x^2 + 3) \) has no real roots.

\( f' \) increases where \( f' > 0 \)

So \( f \) is increasing on \(( -\infty, 0) \cup (4, \infty)\)

4. (15 points) Find the absolute maximum and the absolute minimum values on the closed interval \([0, \frac{3\pi}{2}]\) of the function

\[
f(x) = \sin x + \cos^2 x
\]

\[
f'(x) = \cos(x) + 2 \cos(x)(-\sin(x))
\]

\[
= \cos(x) - 2 \cos(x) \sin(x)
\]

\[
= \cos(x) \left[ 1 - 2 \sin(x) \right]
\]

\( f' \) does not change sign.

\( f' = 0 \Rightarrow \cos(x) = 0 \) or \( 1 - 2 \sin(x) = 0 \)

\( \Rightarrow x = \frac{\pi}{2}, \ \frac{3\pi}{2}, \ \frac{5\pi}{2} \)

\( \Rightarrow \sin(x) = \frac{1}{2} \)

\( \Rightarrow \ x = \frac{\pi}{6}, \ \frac{5\pi}{6} \)

On \([0, \frac{3\pi}{2}]\)

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>\frac{\pi}{6}</td>
<td>\frac{5}{4}</td>
</tr>
<tr>
<td>\frac{\pi}{4}</td>
<td>1</td>
</tr>
<tr>
<td>\frac{5\pi}{6}</td>
<td>\frac{5}{4}</td>
</tr>
<tr>
<td>\frac{3\pi}{4}</td>
<td>-1</td>
</tr>
</tbody>
</table>

Absolute maximum: \( \frac{5}{4} \)

Absolute minimum: \(-1 \)

3.

\[
f(0) = \sin(0) + \left[ \cos(0) \right]^2 = 0 + 1 = 1
\]

\[
f\left( \frac{\pi}{6} \right) = \sin\left( \frac{\pi}{6} \right) + \left[ \cos\left( \frac{\pi}{6} \right) \right]^2 = \frac{1}{2} + \left( \frac{\sqrt{3}}{2} \right)^2 = \frac{1}{2} + \frac{3}{4} = \frac{5}{4}
\]

\[
f\left( \frac{\pi}{4} \right) = \sin\left( \frac{\pi}{4} \right) + \left[ \cos\left( \frac{\pi}{4} \right) \right]^2 = 1 + 0 = 1
\]

\[
f\left( \frac{5\pi}{6} \right) = \sin\left( \frac{5\pi}{6} \right) + \left[ \cos\left( \frac{5\pi}{6} \right) \right]^2 = \frac{1}{2} + \left( \frac{-\sqrt{3}}{2} \right)^2 = \frac{1}{2} + \frac{3}{4} = \frac{5}{4}
\]

\[
f\left( \frac{3\pi}{4} \right) = \sin\left( \frac{3\pi}{4} \right) + \left[ \cos\left( \frac{3\pi}{4} \right) \right]^2 = \frac{1}{2} + \left( \frac{-\sqrt{3}}{2} \right)^2 = \frac{1}{2} + \frac{3}{4} = \frac{5}{4}
\]
5. (15 points) An observer is positioned 3 km away from a rocket launch pad. How fast is the distance between the rocket and the observer increasing, when the rocket is 4 km above the ground and is moving straight up at the speed of 300 m/sec?

\[ \frac{dy}{dt} = 300 \text{ m/s} \]

**Known:** \( \frac{dy}{dt} \)

**Need:** \( \frac{dd}{dt} \) when \( y = 4 \text{ km} = 4000 \text{ m} \).

**Notice:** when \( y = 4000 \text{ m} \), \( d = 5000 \text{ m} \).

Equation relating \( y \) and \( d \):

\[ 3000^2 + y^2 = d^2 \]

\[ \Rightarrow \quad 2y \frac{dy}{dt} = 2d \frac{dd}{dt} \]

\[ \Rightarrow \quad 2(4000)(300) = 2(5000) \frac{dd}{dt} \]

\[ \Rightarrow \quad \frac{dd}{dt} = \frac{2(4000)(300)}{2(5000)} \]

\[ = \frac{4}{5} (300) \]

\[ = 240 \text{ m/s} \]
6. (15 points) Water is leaking from a conical cup at the constant rate of 2 cm³/min. The height of the cup is 12 cm and the radius of the top is 4 cm. How fast is the level of the water in the cup decreasing when the water is 3 cm deep? (The volume of a right circular cone is given by the formula \( V = \frac{1}{3}\pi r^2 h \).)

![Diagram of a conical cup with dimensions and a point at 3 cm depth.]

**KNOWN:** \( \frac{dV}{dt} = -2 \text{ cm}^3/\text{min} \)

NEED \( \frac{dh}{dt} \) when \( h = 3 \)

**Volume:** \( V = \frac{1}{3}\pi r^2 h \)

\[ V = \frac{1}{3}\pi \left(\frac{h}{3}\right)^2 h \]

\[ \Rightarrow V = \frac{1}{27}\pi h^3 \]

So \( \frac{dV}{dt} = \frac{1}{9}\pi h^2 \frac{dh}{dt} \)

\[ -2 = \frac{1}{9}\pi (3)^2 \frac{dh}{dt} \]

\[ -2 = \frac{1}{9}\pi \cdot 9 \frac{dh}{dt} \]

\[ \Rightarrow -\frac{2}{\pi} = \frac{dh}{dt} \]

The water level in the cup is decreasing by \( \frac{2}{\pi} \text{ cm}/\text{min} \) when the water is 3 cm deep.
(35 points) Let \( g(x) = \frac{x^2}{3x-2} \). To save you time, I’m giving you the derivatives of \( g \): \( g'(x) = \frac{3x^2 - 4x}{(3x-2)^2} \) and \( g''(x) = \frac{8}{(3x-2)^3} \).

a. Give the vertical asymptotes. (If there are none, say so.) Remember to justify your answer.

\[
\lim_{x \to \frac{2}{3}^-} \frac{x^2}{3x-2} = -\infty \quad \text{and} \quad \lim_{x \to \frac{2}{3}^+} \frac{x^2}{3x-2} = \infty
\]

So \( x = \frac{2}{3} \) is a vertical asymptote.

b. Give the horizontal asymptotes. (If there are none, say so.) Remember to justify your answer.

\[
\lim_{x \to \infty} \frac{x^2}{3x-2} = \lim_{x \to \infty} \frac{x^2}{x} \frac{1}{3 - \frac{2}{x}} = \lim_{x \to \infty} \frac{x}{3 - \frac{2}{x}} = \infty
\]

\[
\lim_{x \to -\infty} \frac{x^2}{3x-2} = \lim_{x \to -\infty} \frac{x^2}{x} \frac{1}{3 - \frac{2}{x}} = \lim_{x \to -\infty} \frac{x}{3 - \frac{2}{x}} = -\infty
\]

Thus there are no horizontal asymptotes.

Optional observation: \( 3x-2 = \frac{\frac{1}{3}x + \frac{2}{9}}{-\frac{1}{3}x + \frac{2}{9}} \) \( (y = \frac{1}{3}x + \frac{2}{9} \text{ is a slant asymptote}) \)

c. Give the intervals of increasing and decreasing, and give all local maxima and local minima.

\[
g(y_\frac{2}{3}) = \left(\frac{y_\frac{2}{3}}{3(y_\frac{2}{3})-2}\right)^2 = \frac{1 \cdot 9}{1 \cdot 9} = \frac{9}{9} = 1
\]

\[
g'(x) = \frac{3x^2 - 4x}{(3x-2)^2} = \frac{x(3x-4)}{(3x-2)^2}
\]

\[
g' \left( \frac{2}{3} \right) = \frac{\frac{4}{3} \cdot \frac{1}{3}}{\left( -\frac{1}{3} \right)^2} = -\frac{4}{9}
\]

\[
g' \left( \frac{4}{3} \right) = \frac{\frac{16}{3} \cdot \frac{1}{3}}{\left( -\frac{1}{3} \right)^2} = \frac{16}{9}
\]

\[
g \text{ is increasing on } (-\infty, 0) \cup \left( \frac{2}{3}, \frac{4}{3} \right) \cup (\infty, \infty)
\]

\[
g \text{ is decreasing on } (0, \frac{2}{3}) \cup \left( \frac{2}{3}, \frac{4}{3} \right)
\]

Local max: \( (0, 0) \)

Local min: \( \left( \frac{2}{3}, \frac{4}{9} \right) \)

(continued on next page)
d. Give the intervals of concavity and the inflection points.

\[ g''(x) = \frac{8}{(3x-2)^3} \]

\[ g'' \]

\[ \frac{2}{3} \]

\[ g \text{ is concave downward on } (-\infty, \frac{2}{3}) \]

\[ g \text{ is concave upward on } (\frac{2}{3}, \infty) \]

\[ g \text{ changes concavity at } x = \frac{2}{3} \]

but \( \frac{2}{3} \) is not in \( \text{Dom}(g) \),

thus not an inflection point.

e. Sketch the graph of \( g \).