M221 8:00 Sections Test 1 v.1 Fall 2014

No books, no notes, no calculators. Write all your work on this test– nothing else will be graded. You must show your work. Your work must be legible, and your final answers must be reasonably simplified.
On some problems, you are asked to use a specific method to solve the problem (for instance, "Use the definition of the derivative to find..."). On all other problems, you may use any method we've covered. **You may not use methods we have not covered. For instance, use of L'Hopital's Rule will receive no credit.**

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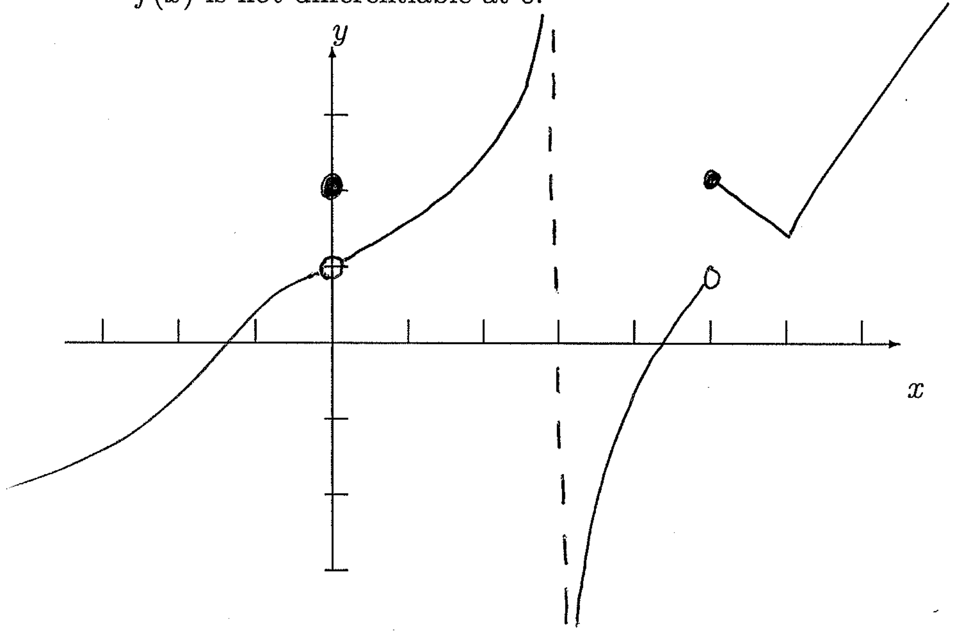
1. (10 points) Draw a graph of a single function $f(x)$ with all of the following properties.
   - the domain of $f(x)$ is $(-\infty, 3) \cup (3, \infty)$.
   - $f(x)$ is not continuous at 0 and is not continuous at 5. It is continuous at all other values in its domain.
   - $\lim_{x \to 0} f(x) = 1$
   - $\lim_{x \to 3^-} f(x) = \infty$
   - $\lim_{x \to 3^+} f(x) = -\infty$
   - $f(x)$ is not differentiable at 6.



2. (10 points) *Working directly from the definition of the derivative,* find $f'(x)$ if

$$f(x) = \sqrt{x+1}.$$

Do not use differentiation formulas or the Chain Rule!

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{x+h+1} - \sqrt{x+1}}{h}$$

$$= \lim_{h\to 0} \frac{\left(\sqrt{x+h+1}-\sqrt{x+1}\right)\left(\sqrt{x+h+1}+\sqrt{x+1}\right)}{h\left(\sqrt{x+h+1}+\sqrt{x+1}\right)}$$

$$= \lim_{h\to 0} \frac{x+h+1-(x+1)}{h\left(\sqrt{x+h+1}+\sqrt{x+1}\right)} = \lim_{h\to 0} \frac{h}{h\left(\sqrt{x+h+1}+\sqrt{x+1}\right)}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{x+h+1}+\sqrt{x+1}} = \frac{1}{2\sqrt{x+1}}$$

3. (12 points) Complete the following statements:

a. A function $f(x)$ is **continuous** at a if:

$$\lim_{x\to a} f(x) = f(a)$$

b. A function $f(x)$ of **differentiable** at a if:

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \text{ exists}$$

c. Which of the following statements is true? Circle a statement if true, cross it out if false.

~~i)~~ If $f(x)$ is continuous at a then $f(x)$ is differentiable at a.

ii) If $f(x)$ is differentiable at a then $f(x)$ is continuous at a.

4. (16 points) Find the following limits. Justify your answer.

a. $\lim_{x\to 3^-} \frac{|x-3|}{x^2-9}$

x is approaching 3 from the left, so $x-3<0$ and $|x-3| = -(x-3)$

$$\lim_{x\to 3^-} \frac{|x-3|}{x^2-9} = \lim_{x\to 3^-} \frac{-(x-3)}{(x-3)(x+3)} = \lim_{x\to 3^-} \frac{-1}{x+3} = -\frac{1}{6}$$

b. $\lim_{x\to 0} \frac{x^2-x}{\sin(2x)}$

$$= \lim_{x\to 0} \frac{x(x-1)}{\sin(2x)} = \lim_{x\to 0} \frac{2}{2}\cdot\frac{x}{\sin(2x)}\cdot(x-1)$$

$$= \left[\lim_{x\to 0} \frac{2x}{\sin(2x)}\right]\cdot\left[\lim_{x\to 0} \frac{1}{2}(x-1)\right] = 1\cdot\left(-\frac{1}{2}\right) = -\frac{1}{2}$$

5. (15 points) A particle is moving along the number line so that its position after t seconds is

$$s(t) = \sin(\pi\sqrt{t}).$$

a) Find its average velocity, in units per second, over the time interval $[0,1]$.

$$\frac{s(1)-s(0)}{1-0} = \frac{\sin(\pi)-\sin(0)}{1} = 0$$

b) Find its instantaneous velocity at time $t=1$.

$$v(t) = s'(t) = \cos(\pi\sqrt{t})\left(\frac{\pi}{2\sqrt{t}}\right)$$

$$v(1) = s'(1) = \cos(\pi)\cdot\left(\frac{\pi}{2}\right) = -\frac{\pi}{2}$$

c) Is the particle speeding up or slowing down at time $t=1$? Justify your answer.

$$a(t) = s''(t) = -\sin(\pi\sqrt{t})\cdot\left(\frac{\pi}{2\sqrt{t}}\right)\left(\frac{\pi}{2\sqrt{t}}\right) + \cos(\pi\sqrt{t})\cdot\left(-\frac{\pi}{4}t^{-\frac{3}{2}}\right)$$

$$a(1) = s''(1) = -\sin(\pi)\cdot\left(\frac{\pi}{2}\right)^2 + \cos(\pi)\left(-\frac{\pi}{4}\right)$$

$$= 0 + \frac{\pi}{4} = \frac{\pi}{4} > 0$$

$v(1)$ is negative and $a(1)$ is positive, hence the particle is slowing down.

6. a. (5 points) Find the critical numbers of the function

$$h(x) = \frac{x+3}{\sqrt{4x+2}}.$$

$$h'(x) = \frac{\sqrt{4x+2} - (x+3)\left(\frac{4}{2\sqrt{4x+2}}\right)}{4x+2} \cdot \frac{\sqrt{4x+2}}{\sqrt{4x+2}}$$

$$= \frac{4x+2-2(x+3)}{(4x+2)^{\frac{3}{2}}} = \frac{2x-4}{(4x+2)^{3/2}}$$

$0 = 2x - 4$
$\Rightarrow 2$ is a critical number.

b. (5 points) Show that the following function is continuous at $x = 0$. (You must clearly show that the definition of continuity holds!)

$$f(x) = \begin{cases} 2x^3 - 3x^2 - 12x + \frac{3}{\sqrt{2}} & \text{if } x \leq 0 \\ \frac{x+3}{\sqrt{4x+2}} & \text{if } x > 0 \end{cases}$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} 2x^3 - 3x^2 - 12x + \frac{3}{\sqrt{2}} = \frac{3}{\sqrt{2}}$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{x+3}{\sqrt{4x+2}} = \frac{3}{\sqrt{2}}$$

$$f(0) = 2\cdot 0^3 - 3\cdot 0^2 - 12\cdot 0 + \frac{3}{\sqrt{2}} = \frac{3}{\sqrt{2}}$$

$\lim_{x\to 0} f(x) = f(0)$, hence $f(x)$ is continuous at 0.

c. (8 points) Find the absolute maximum of the function $f(x)$ above on the interval $[-2, 3]$. (You do not need to find the absolute minimum.)

2 is a critical number (by part a.)

For $x < 0$, $f'(x) = 6x^2 - 6x - 12 = 6(x-2)(x+1)$

Thus -1 is a critical number.

0 may be a critical number.

x	$f(x)$
-2	$-16 - 12 + 24 + \frac{3}{\sqrt{2}} = -4 + \frac{3}{\sqrt{2}}$
-1	$-2 - 3 + 12 + \frac{3}{\sqrt{2}} = 7 + \frac{3}{\sqrt{2}}$
0	$\frac{3}{\sqrt{2}}$
2	$\frac{2+3}{\sqrt{8+2}} = \frac{5}{\sqrt{10}}$
3	$\frac{3+3}{\sqrt{12+2}} = \frac{6}{\sqrt{14}}$

$7 + \frac{3}{\sqrt{2}}$ is the absolute maximum.

7. (5 points) If $3 \le f'(x) \le 5$ for all real numbers x and $f(0) = 1$, which of the following could possibly be true? Circle the ones that could be true and cross out the ones that can't possibly be true. You do not need to justify your answer.

~~a) $f(2) = 0$~~

~~b) $f(2) = 5$~~

c) $f(2) = 10$

~~d) $f(2) = 15$~~

~~e) $f(2) = 20$~~

By the Mean Value Theorem,

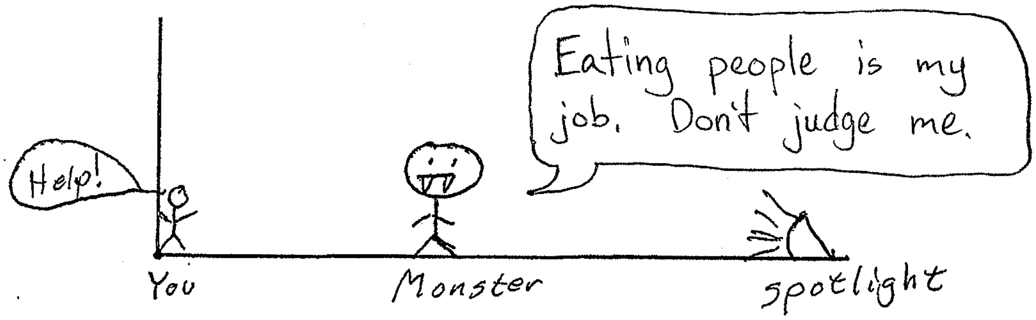
$$f(2) - f(0) \ge 3(2-0)$$

$$f(2) \ge 6 + 1 = 7$$

$$f(2) - f(0) \le 5(2-0)$$

$$f(2) \le 10 + 1 = 11$$

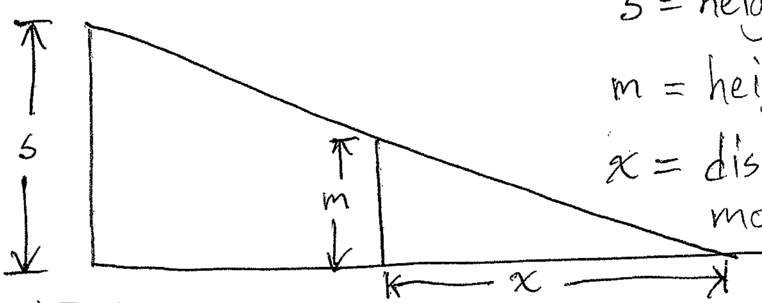
8. (10 points) A horrible green monster with hairy teeth has you trapped against a wall. A spotlight on the ground is shining on this terrible scene, as shown.



The light is 10 meters from the wall. The monster casts a shadow on the wall behind you.

The monster is walking toward you at a speed of 1 meter per second. The monster is also growing at a rate of .5 meters per second. When the monster is 5 meters from the wall, it is 2 meters tall.

a) (2 points) Draw a picture and label the quantities and functions needed to analyze the height of the monster's shadow.



s = height of monster's shadow

m = height of monster

x = distance between the monster and the spotlight

b) (3 points) Find a relationship between the monster's position, height, and the length of its shadow. [Hint: similar triangles.]

$$\frac{s}{10} = \frac{m}{x} \implies sx = 10m$$

c) (4 points) Find the rate at which the monster's shadow is changing when the monster is 5 meters away from the wall.

$$s \cdot (10-5) = 10 \cdot 2 \Rightarrow s = 4$$

$$\frac{d}{dt}(sx) = \frac{d}{dt}(10m)$$

$$\frac{ds}{dt} \cdot x + s \cdot \frac{dx}{dt} = 10\frac{dm}{dt}$$

$$\frac{ds}{dt} \cdot 5 + 4 \cdot 1 = 10 \cdot (.5)$$

$$\frac{ds}{dt} = \frac{5-4}{5} = \frac{1}{5}\ \frac{m}{s}$$

d) (1 points) When the monster is 5 meters from the wall, is his shadow on the wall getting bigger or smaller?

Bigger