1. (18 points) Evaluate each of the following integrals.

a)

$$\int_0^2 \frac{x}{(x^2+1)^2}\,dx = \frac{1}{2}\int_1^5 u^{-2}\,du = -\frac{1}{2}\frac{1}{u}\Big|_1^5$$

$u(0)=1$
$u(2)=5$

$u = x^2+1$
$du = 2x\,dx$
$\frac{du}{2} = x\,dx$

$$= \frac{-1}{10} + \frac{1}{2}$$

$$= \boxed{\frac{2}{5}}$$

b)

$$\int (3x^3 - 6x)(x^2 + x)dx$$

$$= \int 3x^5 - 6x^3 + 3x^4 - 6x^2\,dx$$

$$= \boxed{\frac{x^6}{2} - \frac{3}{2}x^4 + \frac{3}{5}x^5 - 2x^3 + C}$$

c)

$$\int x^2 \sin x\,dx$$

$\int u\,dv = uv - \int v\,du$

$u = x^2 \quad dv = \sin x\,dx$
$du = 2x\,dx \quad v = -\cos x\,dx$

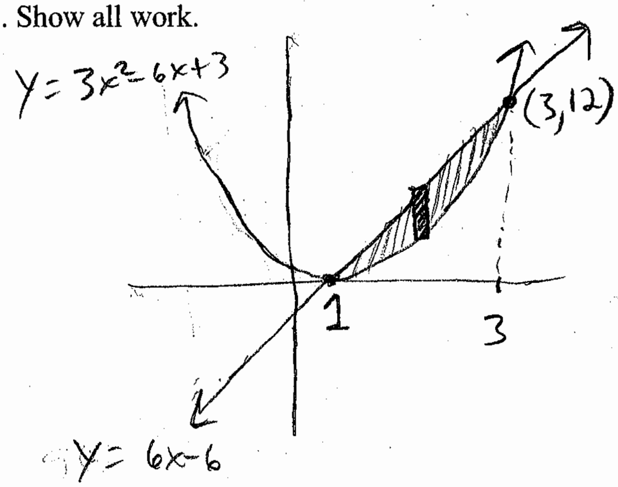
$$= -x^2\cos x + 2\int x\cos x\,dx$$

$u = x \quad dv = \cos x\,dx$
$du = dx \quad v = \sin x$

$$= -x^2\cos x + 2\left[x\sin x - \int \sin x\,dx\right]$$

$$= \boxed{-x^2\cos x + 2x\sin x + 2\cos x + C}$$

2. (12 points) Find the area of the region bounded by the graphs of $f(x) = 3x^2 - 6x + 3$ and $g(x) = 6x - 6$. Show all work.



$3x^2 - 6x + 3 = 6x - 6$

$3x^2 - 12x + 9 = 0$

$x^2 - 4x + 3 = 0$

$(x-3)(x-1) = 0$

$\boxed{x = 1, 3}$

$$\text{Area} = \int_1^3 (6x-6) - (3x^2 - 6x + 3)\,dx$$

$$= \int_1^3 -3x^2 + 12x - 9\,dx$$

$$= \left[\frac{-3x^3}{3} + \frac{12x^2}{2} - 9x\right]_1^3$$

$$= \left[-x^3 + 6x^2 - 9x\right]_1^3$$

$$= \boxed{4}$$

3. (12 points) Suppose $v(t) = t^2 - 6t + 8$, where $t \geq 0$, represents the velocity of a particle where t is in seconds and $v(t)$ is in feet/second.

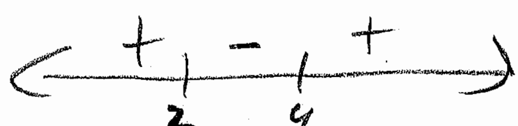
a) Find the average velocity of the particle in the time interval from $t = 1$ to $t = 7$.

$$V_{ave} = \frac{1}{7-1}\int_1^7 t^2 - 6t + 8\,dt$$

$$= \frac{1}{6}\left[\frac{t^3}{3} - 3t^2 + 8t\right]_1^7$$

b) Find the total distance traveled by the particle in the time interval from $t = 1$ to $t = 7$.

$$\int_1^7 |t^2 - 6t + 8|\,dt \Rightarrow |t^2 - 6t + 8| = \begin{cases} (t-4)(t-2), & (t-4)(t-2) \geq 0 \\ -(t-4)(t-2), & (t-4)(t-2) < 0 \end{cases}$$



$$= \int_1^2 t^2 - 6t + 8\,dt - \int_2^4 t^2 - 6t + 8 + \int_4^7 t^2 - 6t + 8\,dt$$

4. (15 points) Find the derivatives specified. You do not need to simplify your answers.

a)

$$\frac{d}{dx}\left[\int_x^5 \frac{t^3+t}{\sqrt{\sin t+2}}\,dt\right] = -\frac{d}{dx}\int_5^x \frac{t^3+t}{\sqrt{\sin t+2}}\,dt$$

$$= -\frac{x^3+x}{\sqrt{\sin x+2}}$$

b)

$$\frac{d}{dx}\left[\int_0^{\sin x} \tan(\sqrt{t})\,dt\right] = \tan(\sqrt{\sin x})\cdot\frac{d}{dx}(\sin x)$$

$$= \cos x\tan(\sqrt{\sin x})$$

c)

$$\frac{d}{dx}\left[\int_{\sqrt{2}}^{\pi} \frac{\sqrt[3]{t}}{t^2+1}\,dt\right] = \boxed{0}$$

5. (10 points) Use calculus to find two nonnegative numbers whose sum is 18 such that the product of one number and the square of the other number is a maximum.

$x+y=18$ $(x, y \geq 0)$

$f(x,y) = x^2 y$ $\boxed{y = 18-x}$

$f(x) = x^2(18-x)$

$f(x) = 18x^2 - x^3$

$f'(x) = 36x - 3x^2 = 0$

$12x - x^2 = 0$

$x(12-x) = 0$

$x = \cancel{0}, 12 \Rightarrow \boxed{x=12} \rightarrow \boxed{y=6}$

Check for max:

$f''(x) = 36 - 6x$

$f''(12) = 36 - 6(12) < 0$, So max @ $x = 12$.

"or" use 1st Derivative:

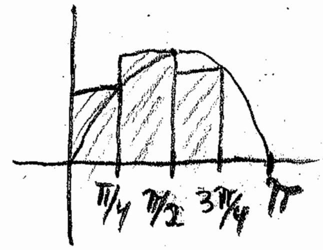
f' $\quad - \quad | \quad + \quad | \quad -$

$\qquad\quad 0 \qquad 12$

6. (15 points) Suppose $f(x) = \sin x$

a) Use a Riemann Sum with four rectangles and right endpoints to estimate the area between $f(x)$ and the x-axis on the interval from $x = 0$ to $x = \pi$.

$$\Delta x = \frac{\pi - 0}{4} = \frac{\pi}{4}$$



$$\text{Area} \approx \sum_{i=1}^{4} f(x_i^*)\Delta x = \frac{\pi}{4}\sum_{i=1}^{4} f(x_i^*)$$

$$= \frac{\pi}{4}\left[f(\pi/4) + f(\pi/2) + f(3\pi/4) + f(\pi)\right]$$

$$= \frac{\pi}{4}\left[\sin(\pi/4) + \sin(\pi/2) + \sin(3\pi/4) + \sin(\pi)\right]$$

$$= \frac{\pi}{4}\left[1/\sqrt{2} + 1 + 1/\sqrt{2} + 0\right] = \boxed{\frac{\pi}{4}(1 + 2/\sqrt{2})}$$

b) Write a limit of sums that gives the exact area under $f(x)$ on $x = 0$ to $x = \pi$. Do not compute (or attempt to compute!) the limit.

$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\frac{\pi}{4} \quad \text{where } x_i^* = 0 + i\left(\frac{\pi}{4}\right)$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \sin\left(\frac{i\pi}{4}\right)\frac{\pi}{4}$$

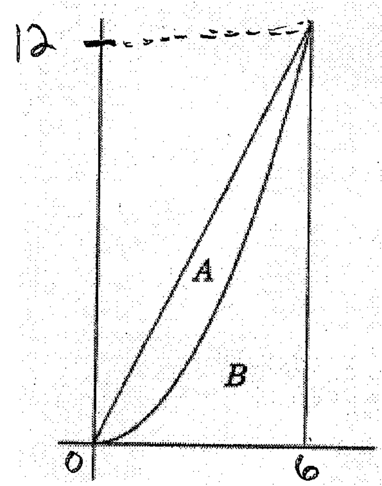
c) Use the Fundamental Theorem of Calculus to find the exact area.

$$\int_0^{\pi} \sin x\,dx = -\cos x\Big|_0^{\pi}$$

$$= -\cos\pi + \cos 0$$

$$= \boxed{2}$$

7. (18 points) Suppose region A is bounded by the curves $y = \frac{x^2}{3}$ and $y = 2x$, and region B is bounded by the curve $y = \frac{x^2}{3}$, the vertical line shown, and the x-axis, as indicated in the graph below:



$$2x = \frac{x^2}{3}$$

$$6x = x^2$$

$$x^2 - 6x = 0$$

$$x(x-6) = 0$$

$$\boxed{x = 0, 6}$$

Set up, but DO NOT EVALUATE, definite integrals for each of the following. Be sure that your answers have the correct limits of integration.

a) The volume of the solid obtained by rotating region B about the x-axis, using disks.

$$V = \pi \int_0^6 \left(\frac{x^2}{3}\right)^2 dx$$

b) The volume of the solid obtained by rotating region A about the y-axis, using cylindrical shells.

$$V = 2\pi \int_0^6 x\left(2x - \frac{x^2}{3}\right) dx$$

c) The volume of the solid obtained by rotating region B about the y-axis, using washers.

$$y = 2x \Rightarrow x = \frac{y}{2} \qquad y = \frac{x^2}{3} \Rightarrow x = \sqrt{3y}$$

$$V = \int_0^{12} 6^2 - (\sqrt{3y})^2 \, dy$$