Math 225 **(White)** **Final Exam**
Differential Calculus **March 14, 2016**

Do NOT turn over this page until instructed to begin.

Name: ____

Section:____

Binghamton ID Number:________________________________

Instructions

Write clear, careful, neat solutions to the questions in the space provided.

No books, no notes, no electronic devices (calculators, cell phones, smart watches, etc.) allowed!

Write all your work on the test–nothing else will be graded. **You must show all your work and justify your answers.** Your work must be legible, and the final answers must be reasonably simplified.

On some problems you may be asked to use a specific method to solve the problem (for instance, "Use the Fundamental Theorem of Calculus to find..."). On all other problems, you may use any method we have covered. **You may not use methods that we have not covered.**

The functions introduced as "black boxes" with given derivatives may be written using either notation, that is, $\exp(x) = e^x$ and $\log(x) = \ln(x)$.

Wandering Eyes Policy

You must keep your eyes on your own work at all times. If you are found looking around, you will be warned once, and only once. A second infraction may result in automatic zero on this test, and possibly a referral to the Harpur College Academic Honesty Committee.

Duration of the Test

This is a timed test designed for one class period. You will start the test when your proctor tells you to start, and you will finish the test when your proctor tells you to stop, when the class period is over.

1	2	3	4	5	6	7	8	Total
20 pts	10 pts	10 pts	15 pts	15 pts	15 pts	8 pts	7 pts	100 pts

1. (20 pts) Evaluate and simplify.

(a) $\displaystyle\int x \ln(x)\ dx$

(b) $\displaystyle\int_0^{\pi/10} x^2 \cos(5x)\ dx$

(c) $\int_0^3 \frac{x^2}{7\sqrt{x^3+1}}\ dx$

(d) $\int_0^1 \sin(\pi\theta + \frac{\pi}{2})\ d\theta$

2. (10 pts)

(a) Find the average value of the function $f(x) = 4 - x^2$ over the interval $[-2, 2]$.

(b) Are there any numbers x in the interval $[-2, 2]$ at which $f(x)$ is equal to its average value? If so, find all such numbers. If not, write “no such numbers exist”.

3. (10 pts) Find the **area** of the region bounded by the following curves:

$$y = x^2 + 4x + 3$$
$$y = -x^2 + 2x + 3$$

Be sure to draw a **picture** and identify and label a typical **slice**.
[Hint: A reasonably accurate picture will help your analysis.]

4. (15 pts) The region R is bounded by the equations $y = 4 - x$ and $y = \frac{1}{2}x^2$ and $x = 0$.

(a) Set up the integral that represents the volume of the solid obtained by rotating the region R about the y-axis. You must choose whether to use disks, washers, or shells, sketch the region, and identify a typical disk or shell.
Do not evaluate the integral.

(b) Set up the integral that represents the volume of the solid obtained by rotating the region R about the x-axis. You must choose whether to use disks, washers, or shells, sketch the region, and identify a typical disk or shell.
Do not evaluate the integral.

5. (15 pts) The region bounded by the curves $y = \sqrt{x}$ and $y = x^3$ is rotated about the line $y = 1$.

(a) Set up the integral that represents the volume of the resulting solid using the **washers** method. You *must* draw a typical washer or otherwise justify your formula.
Do not evaluate the integral.

(b) Set up the integral that represents the volume of the resulting solid using the **shells** method. You *must* draw a typical shell or otherwise justify your formula.
Do not evaluate the integral.

6. (15 pts) You are assigned the job of designing a box. The box must have volume of 225 cubic inches and its base must be rectantular with width 5 times its length. What dimensions should you choose to minimize the the surface (including top and bottom)?
Be sure to draw a picture, identify your symbols, and justify your answer.

7. (8 pts) A stone is thrown vertically upward from the top of a house 60 feet above the ground with an initial velocity of 40 ft/sec. The stone is subject to the pull of gravity, which results in a constant downward acceleration of 32 ft/sec^2.

(a) How long will it take the stone to reach its greatest height?

(b) What is the greatest height reached by the stone?

(c) At what time does the stone pass the roof of the house on the way down?

(d) What is the velocity of the stone at the moment it passes the top of the house on the way down?

8. (7 pts) The velocity of a particle is given by $v(t) = \sec^2(t) + 2\cos(2t)$, starting at time $t = 0$. After $\pi/4$ seconds, how far has the particle travelled from its initial position?

Math 225
Differential Calculus

(White)

Final Exam
March 14, 2016

Do NOT turn over this page until instructed to begin.

Name: ______________________________

Section: ______________________________

Binghamton ID Number: ______________________________

Instructions

Write clear, careful, neat solutions to the questions in the space provided.

No books, no notes, no electronic devices (calculators, cell phones, smart watches, etc.) allowed!

Write all your work on the test–nothing else will be graded. **You must show all your work and justify your answers.** Your work must be legible, and the final answers must be reasonably simplified.

On some problems you may be asked to use a specific method to solve the problem (for instance, "Use the Fundamental Theorem of Calculus to find..."). On all other problems, you may use any method we have covered. **You may not use methods that we have not covered.**

The functions introduced as "black boxes" with given derivatives may be written using either notation, that is, $\exp(x) = e^x$ and $\log(x) = \ln(x)$.

Wandering Eyes Policy

You must keep your eyes on your own work at all times. If you are found looking around, you will be warned once, and only once. A second infraction may result in automatic zero on this test, and possibly a referral to the Harpur College Academic Honesty Committee.

Duration of the Test

This is a timed test designed for one class period. You will start the test when your proctor tells you to start, and you will finish the test when your proctor tells you to stop, when the class period is over.

1	2	3	4	5	6	7	8	Total
20 pts	10 pts	10 pts	15 pts	15 pts	15 pts	8 pts	7 pts	100 pts

1. (20 pts) Evaluate and simplify.

(a) $\int x \ln(x)\ dx$

Integration "by parts"

$u(x) = \ln(x) \qquad v'(x) = x$

$u'(x) = \frac{1}{x} \qquad v(x) = \frac{1}{2}x^2$

$= \frac{1}{2}x^2 \ln(x) - \int \left(\frac{1}{x}\right)\left(\frac{1}{2}x^2\right) dx$

$= \frac{1}{2}x^2 \ln(x) - \frac{1}{2}\int x\, dx$

$= \frac{1}{2}x^2 \ln(x) - \frac{1}{2}\left(\frac{1}{2}x^2\right) + C$

$= \frac{1}{2}x^2 \ln(x) - \frac{1}{4}x^2 + C$

(b) $\int_0^{\pi/10} x^2 \cos(5x)\ dx$

Integration "by parts".

$u(x) = x^2 \qquad v'(x) = \cos(5x)$

$u'(x) = 2x \qquad v(x) = \frac{1}{5}\sin(5x)$

Redefine u, v

$u(x) = x \qquad v'(x) = \sin(5x)$

$u'(x) = 1 \qquad v(x) = \frac{-1}{5}\cos(5x)$

$= \frac{1}{5}x^2\sin(5x)\Big]_0^{\pi/10} - \int_0^{\pi/10} \frac{2}{5}x\sin(5x)\,dx$

$= \frac{1}{5}x^2\sin(5x)\Big]_0^{\pi/10} - \frac{2}{5}\int_0^{\pi/10} x\sin(5x)\,dx$

$= \frac{1}{5}x^2\sin(5x)\Big]_0^{\pi/10} - \frac{2}{5}\left[-\frac{1}{5}x\cos(5x)\Big]_0^{\pi/10} - \int_0^{\pi/10} \frac{-1}{5}\cos(5x)\,dx\right]$

$= \frac{1}{5}x^2\sin(5x)\Big]_0^{\pi/10} + \frac{2}{25}x\cos(5x)\Big]_0^{\pi/10} - \frac{2}{25}\int_0^{\pi/10}\cos(5x)\,dx$

$= \frac{1}{5}x^2\sin(5x)\Big]_0^{\pi/10} + \frac{2}{25}x\cos(5x)\Big]_0^{\pi/10} - \frac{2}{25}\cdot\frac{1}{5}\sin(5x)\Big]_0^{\pi/10}$

$= \left(\frac{1}{5}\cdot\frac{\pi^2}{100}\cdot\sin\left(\frac{\pi}{2}\right) - 0\right) + \left(\frac{2}{25}\cdot\frac{\pi}{10}\cos\left(\frac{\pi}{2}\right) - 0\right) - \left(\frac{2}{125}\sin\left(\frac{\pi}{2}\right) - 0\right)$

$= \frac{\pi^2}{500} + 0 - \frac{2}{125}$

$= \frac{\pi^2}{500} - \frac{2}{125}$

$= \frac{1}{500}\left(\pi^2 - 8\right)$

(c) $\int_0^3 \frac{x^2}{7\sqrt{x^3+1}}\,dx$

Substitution: $u(x) = x^3+1$

$u'(x) = 3x^2$

$$= \int_0^3 \frac{3x^2}{21\sqrt{x^3+1}}\,dx$$

$$= \int_0^3 \frac{u'(x)}{21\sqrt{u(x)}}\,dx$$

$$= \frac{1}{21}\int_0^3 (u(x))^{-\frac{1}{2}}\,u'(x)\,dx$$

$$= \frac{1}{21}\int_1^{28} u^{-\frac{1}{2}}\,du$$

$$= \frac{1}{21}\left(2u^{\frac{1}{2}}\right)\Big|_1^{28}$$

$$= \frac{2}{21}\left(\sqrt{28}-\sqrt{1}\right) = \frac{2}{21}\left(\sqrt{28}-1\right)$$

(d) $\int_0^1 \sin(\pi\theta + \frac{\pi}{2})\,d\theta$

Substitution: $u(\theta) = \pi\theta + \frac{\pi}{2}$

$u'(\theta) = \pi$

$$= \int_0^1 \frac{1}{\pi}\sin(u(\theta))\,u'(\theta)\,d\theta$$

$$= \frac{1}{\pi}\int_0^1 \sin(u(\theta))\,u'(\theta)\,d\theta$$

$$= \frac{1}{\pi}\int_{\pi/2}^{3\pi/2} \sin(u)\,du$$

$$= \frac{1}{\pi}\left(-\cos(u)\Big|_{\pi/2}^{3\pi/2}\right)$$

$$= \frac{1}{\pi}\left(-\cos\left(\frac{3\pi}{2}\right) + \cos\left(\frac{\pi}{2}\right)\right)$$

$$= \frac{1}{\pi}(0+0) = 0$$

2. (10 pts)

(a) Find the average value of the function $f(x) = 4 - x^2$ over the interval $[-2, 2]$.

$$f_{avg} = \frac{1}{2-(-2)} \int_{-2}^{2} 4 - x^2 \, dx$$

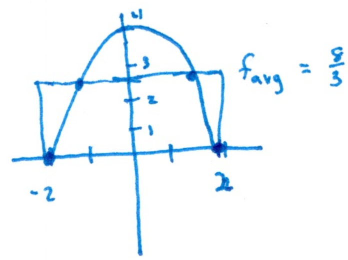
$$= \frac{1}{4}\left[4x - \frac{1}{3}x^3\right]\Big|_{-2}^{2}$$

$$= \frac{1}{4}\left[\left(8 - \frac{8}{3}\right) - \left(-8 - \frac{(-8)}{3}\right)\right]$$

$$= \frac{1}{4}\left[\frac{16}{3} + \frac{16}{3}\right]$$

$$= \frac{32}{12}$$

$$= \frac{8}{3}$$



(b) Are there any numbers x in the interval $[-2, 2]$ at which $f(x)$ is equal to its average value? If so, find all such numbers. If not, write "no such numbers exist".

We wish to find a in $[-2, 2]$ such that

$f(a) = \frac{8}{3}$, i.e.,

$$4 - a^2 = \frac{8}{3}$$

$$a^2 = 4 - \frac{8}{3}$$

$$= \frac{12}{3} - \frac{8}{3}$$

$$= \frac{4}{3}$$

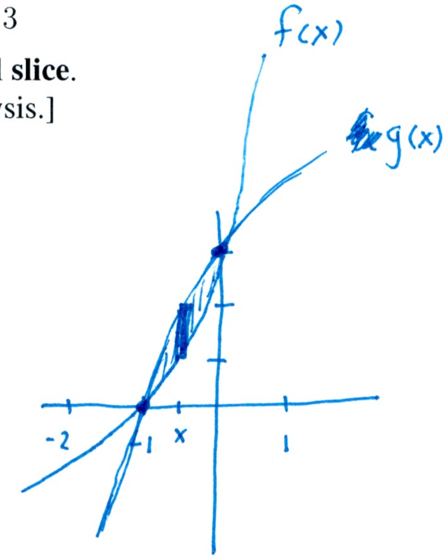
$a_1 = \frac{2}{\sqrt{3}}$ and $a_2 = \frac{-2}{\sqrt{3}}$ are both solutions.

Since $\sqrt{3} > 1$, $\frac{2}{\sqrt{3}} < 2$ and $\frac{-2}{\sqrt{3}} > -2$, so both values lie in the interval $[-2, 2]$.

3. (10 pts) Find the **area** of the region bounded by the following curves:

$$y = x^2 + 4x + 3$$
$$y = -x^2 + 2x + 3$$

Be sure to draw a **picture** and identify and label a typical **slice**.
[Hint: A reasonably accurate picture will help your analysis.]



Let $f(x) = x^2 + 4x + 3$
$g(x) = -x^2 + 2x + 3$.

Then $f(x) = g(x)$ when

$x^2 + 4x + 3 = -x^2 + 2x + 3$

$2x^2 + 2x = 0$

$\quad \| $

$2x(x+1)$

Curves intersect when $x = 0$ and $x = -1$

$f(0) = g(0) = 3$

$f(-1) = g(-1) = 0$

Take x in $[-1, 0]$.

Area of slice at x :

$g(x) - f(x) = (-x^2+2x+3) - (x^2+4x+3)$

$= -2x^2 - 2x$

$(-2x^2 - 2x)\Delta x$

Area of region: $\displaystyle\int_{-1}^{0} (-2x^2 - 2x)\,dx$

$= -2 \displaystyle\int_{-1}^{0} x^2 + x \, dx$

$= -2 \left[\dfrac{x^3}{3} + \dfrac{x^2}{2} \Big|_{-1}^{0} \right]$

$= -2 \left[0 - \left(-\dfrac{1}{3} + \dfrac{1}{2} \right) \right]$

$= -2 \left(\dfrac{1}{3} - \dfrac{1}{2} \right)$

$= -\dfrac{2}{3} + 1$

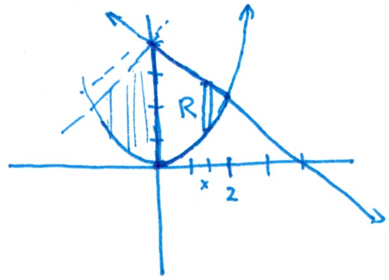
$= \dfrac{1}{3}$

4. (15 pts) The region R is bounded by the equations $y = 4 - x$ and $y = \frac{1}{2}x^2$ and $x = 0$.

(a) Set up the integral that represents the volume of the solid obtained by rotating the region R about the y-axis. You must choose whether to use disks, washers, or shells, sketch the region, and identify a typical disk or shell.
Do not evaluate the integral.

Choose x in $[0, 2]$.

Shell:



$$V = \int_0^2 2\pi x\left(4 - x - \frac{1}{2}x^2\right) dx$$

$\frac{1}{2}x^2 = 4 - x$

$x^2 = 8 - 2x$

$x^2 + 2x - 8 = 0$

$(x+4)(x-2)$

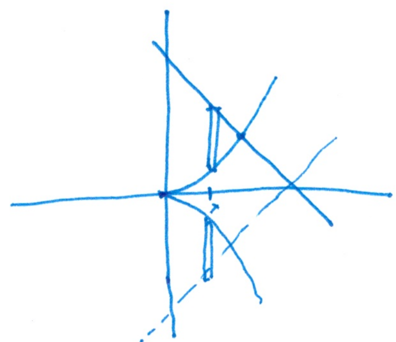
(b) Set up the integral that represents the volume of the solid obtained by rotating the region R about the x-axis. You must choose whether to use disks, washers, or shells, sketch the region, and identify a typical disk or shell.
Do not evaluate the integral.

Choose x in $[0, 2]$

Disk:

$\pi(\text{outside})^2 - \pi(\text{inside})^2$

$= \pi(4-x)^2 - \pi\left(\frac{1}{2}x^2\right)^2$



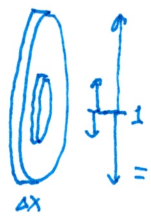
$$V = \int_0^2 \pi(4-x)^2 - \pi\left(\frac{1}{2}x^2\right)^2 dx$$

5. (15 pts) The region bounded by the curves $y = \sqrt{x}$ and $y = x^3$ is rotated about the line $y = 1$.

(a) Set up the integral that represents the volume of the resulting solid using the **washers** method. You *must* draw a typical washer or otherwise justify your formula.
Do not evaluate the integral.

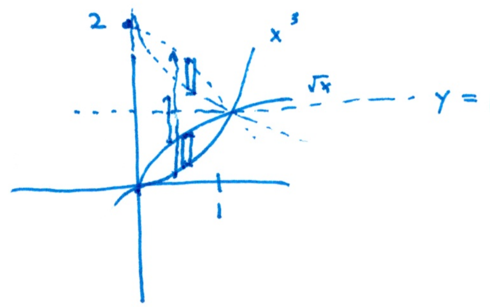
Choose x in $[0,1]$.

Washer



$\pi(\text{outer})^2 - \pi(\text{inner})^2$

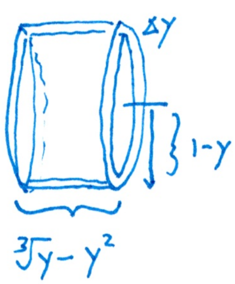
$= \pi(1-x^3)^2 - \pi(1-\sqrt{x})^2$

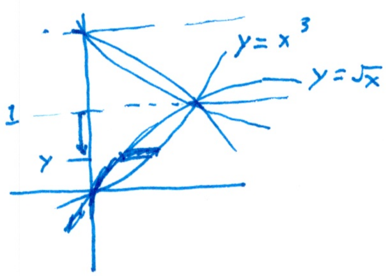


$$V = \int_0^1 \pi(1-x^3)^2 - \pi(1-\sqrt{x})^2\, dx$$

(b) Set up the integral that represents the volume of the resulting solid using the **shells** method. You *must* draw a typical shell or otherwise justify your formula.
Do not evaluate the integral.

Choose y in $[0,1]$

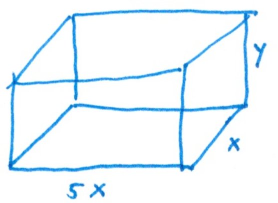




$$V = \int_0^1 2\pi(1-y)(\sqrt[3]{y} - y^2)\, dy$$

rectangular

6. (15 pts) You are assigned the job of designing a box. The box must have volume of 225 cubic inches and its base must be rectantular with width 5 times its length. What dimensions should you choose to minimize the the surface (including top and bottom)?
Be sure to draw a picture, identify your symbols, and justify your answer.



Let x be the length, so $5x$ width.
Let y be the height.
Let V be the volume
let S be the surface area

Know $V = (5x)(x)(y) = 225$ volume

$$S = 2(5x)(x) + 2(x)(y) + 2(5x)y$$
$$= 10x^2 + 2xy + 10xy$$
$$= 10x^2 + 12xy \quad \text{surface area}$$

Then $5x^2y = 225$

$$y = \frac{225}{5x^2} = \frac{45}{x^2}$$

Wish to minimize $S(x) = 10x^2 + 12x\left(\frac{45}{x^2}\right)$

$$= 10x^2 + \frac{(12)(45)}{x}$$

over domain $(0, \infty)$.

$S''(x) = 20 + \frac{2(12)(45)}{x^3}$

Since $S''(x)$ is positive on $(0,\infty)$, the c.p. at 3 is a minimum,

Dimensions $3, 15, \frac{45}{9} = 5$ inches

Extrema occur at critical pts:

$$S'(x) = 20x - \frac{(12)(45)}{x^2}$$

$S'(x) = 0$ when $20x - \frac{(12)(45)}{x^2} = 0$

$$20x^3 - (12)(45) = 0$$

$$x^3 = \frac{(12)(45)}{20} = 3 \cdot 9 = 3^3$$

$$x = 3$$

Only c.p. on the domain is $x = 3$.

7. (8 pts) A stone is thrown vertically upward from the top of a house 60 feet above the ground with an initial velocity of 40 ft/sec. The stone is subject to the pull of gravity, which results in a constant downward acceleration of 32 ft/sec^2.

(a) How long will it take the stone to reach its greatest height?

$a(t) = -32$

$v(t) = -32t + C$

$40 = v(0) = 0 + C$

$v(t) = -32t + 40$

The greatest height occurs when $v(t) = 0$.

$v(t_{top}) = 0 = -32\,t_{top} + 40$

$32\,t_{top} = 40$

$t_{top} = \frac{40}{32} = \frac{5}{4}$ seconds.

(b) What is the greatest height reached by the stone?

$s(t) = -32 \cdot \frac{1}{2} t^2 + 40t + d$

$60 = s(0) = d$

$s(t) = -16t^2 + 40t + 60$

$s(\frac{5}{4}) = -16(\frac{5}{4})^2 + 40(\frac{5}{4}) + 60$

$= -25 + 50 + 60 = 85$ feet.

(c) At what time does the stone pass the roof of the house on the way down?

$60 = s(t_{roof}) = -16t_{roof}^2 + 40t_{roof} + 60$

$-16t_{roof}^2 + 40t_{roof} = 0$

$\parallel$

$-8t_{roof}(2t_{roof} - 5)$

Solutions:

$t_{roof} = 0$ or $t_{roof} = \frac{5}{2}$ seconds.

On the way down the time is

$t_{roof} = \frac{5}{2}$ seconds

(d) What is the velocity of the stone at the moment it passes the top of the house on the way down?

$v(\frac{5}{2}) = -32(\frac{5}{2}) + 40 = (-16)(5) + 40 = -80 + 40 = -40$ ft/sec.

8. (7 pts) The velocity of a particle is given by $v(t) = \sec^2(t) + 2\cos(2t)$, starting at time $t = 0$. After $\pi/4$ seconds, how far has the particle travelled from its initial position?

$$\int_0^{\pi/4} \sec^2(t) + 2\cos(2t)\, dt$$

$$= \tan(t)\Big]_0^{\pi/4} + 2 \cdot \tfrac{1}{2} \sin(2t)\Big]_0^{\pi/4}$$

$$= \left[\tan\left(\tfrac{\pi}{4}\right) - \tan(0)\right] + \left(\sin\left(\tfrac{\pi}{2}\right) - \sin(0)\right)$$

$$= (1 - 0) + (1 - 0)$$

$$= 2.$$