1. [10]	2. [15]	3. [10]	4. [10]	5. [5]	6. [10]	7. [5]	8. [10]	9.[25]	Total: [100]

Math 224 **Final V.1 (White)**

Name: Solutions *Section:*

Closed book and closed notes. *Answers must include supporting work.*

Calculators and cell phones out of sight.

1. (10 pts) Find the equation of the tangent line to the curve at the given point.

$$x^2y - xy^2 = -6, \quad (2,3)$$

$$\frac{d}{dx}(x^2y - xy^2) = \frac{d}{dx}(-6)$$

$$2xy + x^2y' - y^2 - 2xyy' = 0$$

$$x^2y' - 2xyy' = y^2 - 2xy$$

$$y'(x^2 - 2xy) = y^2 - 2xy$$

$$y' = \frac{y^2 - 2xy}{x^2 - 2xy}$$

$$y'\Big|_{(2,3)} = \frac{9 - 2(6)}{4 - 2(6)} = \frac{9-12}{4-12} = \boxed{\frac{3}{8}} = m$$

$$Y_T = mx + b$$

$$Y_T = \frac{3}{8}x + b$$

Plug in (2,3):

$$3 = \frac{3}{8}(2) + b \Rightarrow b = 3 - \frac{6}{8} = \frac{9}{4}$$

$$\boxed{Y_T = \frac{3}{8}x + \frac{9}{4}}$$

2. (15 pts) Determine the following limits and show all steps in calculating them. If the limit doesn't exist because it's positive or negative infinity, then say so.

a) $\lim_{x\to\infty} \frac{17x^5 - 4x^2 + 10}{6x^6 + 5x^3 + x} \cdot \frac{1/x^6}{1/x^6} = \lim_{x\to\infty} \frac{17/x - 4/x^4 + \frac{10}{x^6}}{6 + 5/x^3 + 1/x^5}$

$= \boxed{0}$

b) $\lim_{x\to\infty}\left(\sqrt{9x^2 - x} - 3x\right) \frac{\sqrt{9x^2-x} + 3x}{\sqrt{9x^2-x} + 3x} = \lim_{x\to\infty} \frac{(9x^2-x) - 9x^2}{\sqrt{9x^2-x} + 3x}$

$= \lim_{x\to\infty} \frac{-x}{\sqrt{9x^2-x} + 3x} \cdot \frac{1/\sqrt{x^2}}{1/\sqrt{x^2}}$

$= \lim_{x\to\infty} \frac{-1}{\sqrt{9 - 1/x} + 3}$

$= \boxed{\frac{-1}{6}}$

d) $\lim_{x\to-\infty} \frac{\sqrt{4x^2 - 5x + 3}}{3x - 7} \cdot \frac{1/\sqrt{x^2}}{1/\sqrt{x^2}}$

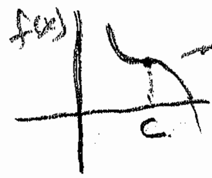
$= \lim_{x\to-\infty} \frac{\sqrt{4 - 5/x + 3/x^2}}{-3 + 7/x}$

$= \boxed{\frac{-\sqrt{4}}{3}}$

3. (10 pts) Assume that the function $f(x)$ is continuous and differentiable for all real numbers. Answer whether the following statements are "True" or "False". If the answer is "False", then give an example where the statement is false or explain why it's false.

a) If $f'(c) = 0$, then f has a local maximum or local minimum value at $x = c$.

"False"



→ $f'(c) = 0$ with no local max/min

"or" $f(x) = x^3$ @ $x = 0$

b) If $f''(a) = 0$, then f has a point of inflection at $x = a$.

"False"

$f(x) = x^4$ @ $x = 0$

c) It's possible for $f(x)$ to be both increasing and $f''(x) < 0$.

True

d) $f(x)$ is guaranteed to have an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a,b]$.

True

e) If $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} g(x) = \infty$ then $\lim_{x \to a} [f(x) - g(x)] = 0$.

"False"

$$\lim_{x \to 0} \left[\frac{1}{x^2} - \frac{1}{x}\right] = \lim_{x \to 0} \left(\frac{1-x^2}{x^2}\right) = \infty$$

4. (10 pts) Find the absolute maximum and minimum values of $f(x) = x\sqrt{8-x^2}$ on the interval $\left[-1, 2\sqrt{2}\right]$.

$f(x) = x(8-x^2)^{1/2}$

$f'(x) = (1)(8-x^2)^{1/2} + \frac{1}{2}x(8-x^2)^{-1/2}(-2x)$

$$f'(x) = \sqrt{8-x^2} - \frac{x^2}{\sqrt{8-x^2}} = 0$$

$8 - x^2 - x^2 = 0$

$2x^2 = 8$

$\boxed{x = \pm 2}$ $x = -2$ not in $[-1, 2\sqrt{2}]$

so jus $\boxed{x = 2}$

Check pts. $x = -1, 2, 2\sqrt{2}$:

$f(-1) = \boxed{-\sqrt{7}}$ Abs. min

$f(2) = \boxed{4}$ Abs. max

$f(2\sqrt{2}) = 0$

5. (5 pts) Find the vertical and horizontal asymptotes of the curve $y = \dfrac{2x^2 + x - 1}{x^2 - x - 2}$.

V.A.

$$y = \frac{(2x-1)(x+1)}{(x-2)(x+1)} = \frac{2x-1}{x-2} \text{ when } x \neq -1$$

$$\lim_{x \to -1} \frac{2x-1}{x-2} = \frac{-3}{-3} = -1$$

$$\lim_{x \to 2^+} \frac{2x-1}{x-2} = \infty$$

$\boxed{\text{V.A. @ } x = 2}$

H.A.

$$\lim_{x \to \pm\infty} \frac{2x^2 + x - 1}{x^2 - x - 2} = 2$$

$\boxed{\text{H.A. @ } y = 2}$

6. (10 pts) The radius of a sphere is increasing at a rate of 20 mm/s. How fast is the volume increasing when the radius is 30 mm? (The volume of a sphere is $V = \frac{4}{3}\pi r^3$)

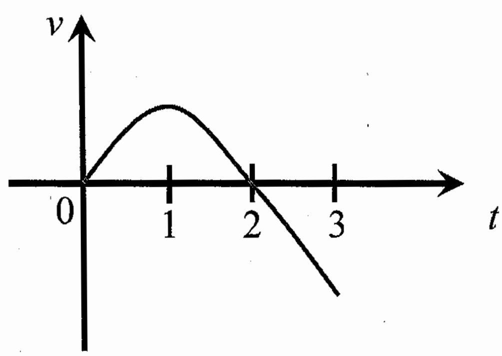
What is $\frac{dV}{dt}$ when $r = 30$ and $\frac{dr}{dt} = 20$?

$$\frac{dV}{dt} = \frac{4}{3}\pi(3)r^2\frac{dr}{dt}$$

$$\frac{dV}{dt} = 4\pi(30)^2(20) \text{ mm}^3/\text{s}$$

$$= 72{,}000\pi \text{ mm}^3/\text{s}$$

7. (5 points) The graph of the velocity function of a particle is shown, where t is measured in seconds. When is the particle speeding up? When is it slowing down? Explain.



$v(t) = 1-(t-1)^2$ for $0 \le t \le 3$

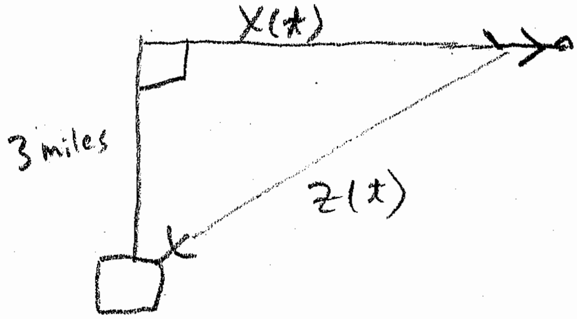
V(t) + + −

a(t) + 1 − 2 − 3

Speeding up on $(0,1) \cup (2,3]$

Slowing down on $(1,2)$

8. (10 pts) A plane flying horizontally at an altitude of 3 miles and a velocity of 300 mi/hr passes directly over a radar station. Find the rate at which the straight line distance from the plane to the station is increasing when the plane is 5 miles away from the station.



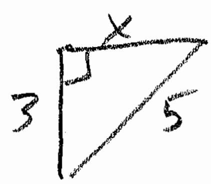
$\rightarrow \frac{dx}{dt} = 300 \text{ m/h}$

What is $\frac{dz}{dt}$ when $z = 5$?

$$3^2 + x^2 = z^2$$

$$2x\frac{dx}{dt} = 2z\frac{dz}{dt}$$

when $z = 5$:



$\Rightarrow 9 + x^2 = 25$

$x^2 = 16$

$x = \pm 4 \Rightarrow x = 4$

Plug in $z=5$, $x = 4$ and $\frac{dx}{dt} = 300$:

$$2(4)(300) = 2(5)\frac{dz}{dt}$$

$$\boxed{\frac{dz}{dt} = 240 \text{ m/h}}$$

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* Students might interpret $x=5$ instead of $z=5$.

i.e. "What is $\frac{dz}{dt}$ when $x = 5$?"

In this case $z = \sqrt{34}$ and $\boxed{\frac{dz}{dt} = \frac{1500}{\sqrt{34}} \text{ m/h}}$.

9. (25 pts) The function is given by $f(x) = \frac{x}{x^2+4}$.

Its derivative and second derivative are given by

$$f'(x) = \frac{4-x^2}{\left(x^2+4\right)^2} \qquad f''(x) = \frac{2x(x^2-12)}{\left(x^2+4\right)^3}.$$

Find the following for $f(x)$:

a) Domain

All reals

b) The roots (i.e. x-intercepts)

$\frac{x}{x^2+4} = 0 \Rightarrow \boxed{x=0}$

Notice f: $-$ | $+$ at 0; below x-axis, above x-axis

c) Vertical and horizontal asymptotes

No V.A.

H.A. @ $y=0$

d) Intervals increasing/decreasing

$f'(x) = \frac{4-x^2}{(x^2+4)^2} = 0 \Rightarrow 4-x^2=0$

$= \frac{-(x-2)(x+2)}{(x^2+4)^2}$

$\boxed{x = \pm 2}$ crit. #'s

f': $-$ | $+$ | $-$ at -2, 2

f: dec, -2, inc, 2, dec

min max

e) Local maximum and minimum values

Local min @ $x=-2$, Local max @ $x=2$

$f(-2) = \frac{-2}{8} = \boxed{-\frac{1}{4}}$ Local min value

$f(2) = \boxed{\frac{1}{4}}$ Local max value

f) Intervals of concavity and points of inflection

$$f'' = \frac{2x(x^2-12)}{(x^2+4)^3} = 0 \Rightarrow x=0 \text{ or } x^2-12=0$$

$$x = \pm 2\sqrt{3}$$

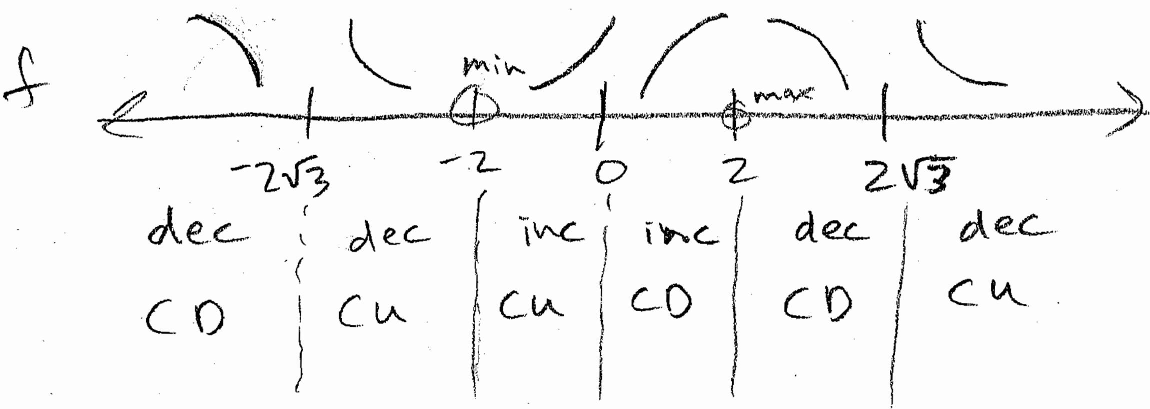
Possible pts. of inflection

f'': $-$ | $-2\sqrt{3}$ | $+$ | 0 | $-$ | $2\sqrt{3}$ | $+$

CD, CU, CD, CU

$\Rightarrow$ pts. of inflection @ $x = 0, \pm 2\sqrt{3}$

g) Sketch



pts. of inflection:
$(0,0)$
$\left(2\sqrt{3}, \frac{\sqrt{3}}{5}\right)$
$\left(-2\sqrt{3}, \frac{-\sqrt{3}}{5}\right)$