Fall 2022 solutions | Marwa Mosallam | 1

1) a) using your unit circle and since cosine is an even function, we know that

$$\cos\left(-\frac{5\pi}{6}\right) = \cos\left(\frac{5\pi}{6}\right) = +\frac{\sqrt{3}}{2}$$

Also, note that 5π is equivalent to π as every 2π the function repeats itself

b) $\csc\left(\frac{\pi}{4}\right) = \frac{1}{\sin(\pi/4)} = \frac{1}{1/\sqrt{2}} = \sqrt{2}$

note that $\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$.

c) $\tan\left(\frac{9\pi}{2}\right) = \tan\left(\frac{\pi}{2}\right)$ which is undefined

Note that $9\pi = 2\pi + 2\pi + 2\pi + 2\pi + \pi$ but since the period of tan is 2π so we can ignore 8π and the remainder is π

2) Expanding $\log_3 \frac{(x+1)\sqrt{x-7}}{9(x-4)^2}$

solution. By properties of logarithmic funct. we know, $\log_3 \frac{x}{y} = \log_3 x - \log_3 y$

Then $\log_3 \frac{(x+1)\sqrt{x-7}}{9(x-4)^2} = \log_3\left((x+1)\sqrt{x-7}\right) - \log_3\left(9(x-4)^2\right)$

$\longrightarrow (1.)$

Also, by properties of logarithmic functions we know

$$\log_3 xy = \log_3 x + \log_3 y$$

Then, from (1.), we have

$$\log_3 \frac{(x+1)\sqrt{x-7}}{9(x-4)^2} = \log_3(x+1) + \log_3(x-7)^{1/2} - \log_3 9 - 2\log_3(x-4)$$

$$= \log_3(x+1) + \frac{1}{2}\log_3(x-7) - 2 - 2\log_3(x-4).$$

3) (a)

let $\tan^{-1}(\sqrt{3}) = x$, then $\sqrt{3} = \tan x$, which means $\frac{\sin x}{\cos x} = \frac{\sqrt{3}}{1} = \frac{\sqrt{3}/2}{1/2}$

Then from the unit circle

$$x = \frac{\pi}{3} \text{ or } x = \frac{4\pi}{3}$$

(b) $\arcsin(-1) = \sin^{-1}(-1)$.

Let $\sin^{-1}(-1) = x$, then $-1 = \sin x$, then from the drawing of the sine function, $x = \frac{3\pi}{2}$.

(c) $5^{\log_5 9}$.

we know that the exponential function is the inverse of the logarithmic function then 5 & $\log_5$

can be cancelled with each other and we get

$$5^{\log_5 9} = 9$$

(d) $\ln\left(\frac{1}{e^2}\right) = \ln e^{-2} = -2\ln e$

$= -2\log_e e$

$= -2$

because $\log_e e = 1$

(e) $\arccos\left(\cos\left(-\frac{\pi}{3}\right)\right) = \cos^{-1}\left(\cos\left(-\frac{\pi}{3}\right)\right)$

$= -\frac{\pi}{3}$

as $\cos^{-1}$ & $\cos$ are inverse functions of each other and so they are cancelled with each other

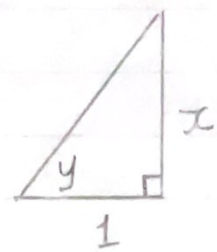
4) $\sin(\tan^{-1}(x))$ as an algebraic expression.

solution.

let $\tan^{-1}x = y$, then

$x = \tan y$

And we can draw the following triangle



and by pythagorus, we know that the hypotenus is

$$\sqrt{1+x^2}$$

Now, we want to know $\sin(y)$, then from the triangle above

$$\sin(y) = \frac{x}{\sqrt{1+x^2}}$$

5) Solve each equation.

(a) $\ln(x^2-1)=0$

solution. take the exponential of both sides

$$e^{\ln(x^2-1)} = e^0$$

But $e^0=1$ & exponential is the inverse of ln so they cancel, so we have

$$x^2-1 = 1$$

$$x^2=2$$

take the square root of both side of the previous eqn., we get

$$|x| = \sqrt{2}$$

hence $\quad x = \pm\sqrt{2}$

(b) $e^{2x-3} = 12$

Solution.

take the ln of both sides

$$\ln e^{2x-3} = \ln 12 \longrightarrow (1.)$$

$\ln = \log_e$, then $\ln$ & e in (1.) cancels, hence

$$2x - 3 = \ln(12)$$
$$2x = (\ln(12)) + 3$$
$$x = (\ln(12)) + 3 \,/\, 2$$

(c) $3\sec^2(x) - 4 = 0$ over the interval $[0, 2\pi]$.

solution.

$$3\sec^2(x) \cancel{-4} = 0$$
$$\cancel{+4} \quad +4$$
$$3\sec^2(x) = 4$$
$$\sec^2(x) = 4/3$$

then $\frac{1}{\cos^2(x)} = \frac{4}{3}$, so $\cos^2(x) = \frac{3}{4} \rightarrow (1.)$

taking square root of both sides of (1.), we get

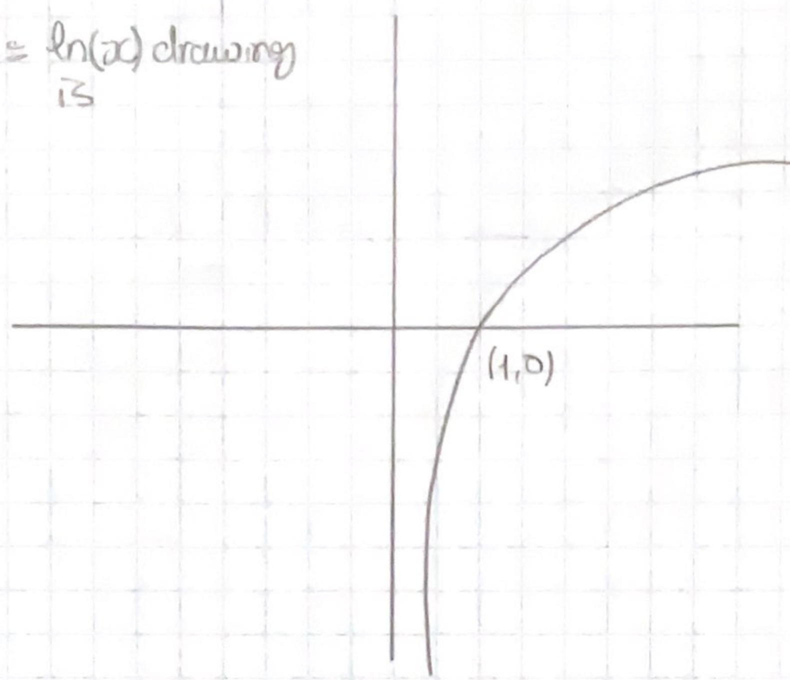
$$\cos(x) = \pm \frac{\sqrt{3}}{2}$$

then $x = \frac{\pi}{6}$ or $x = \frac{7\pi}{6}$

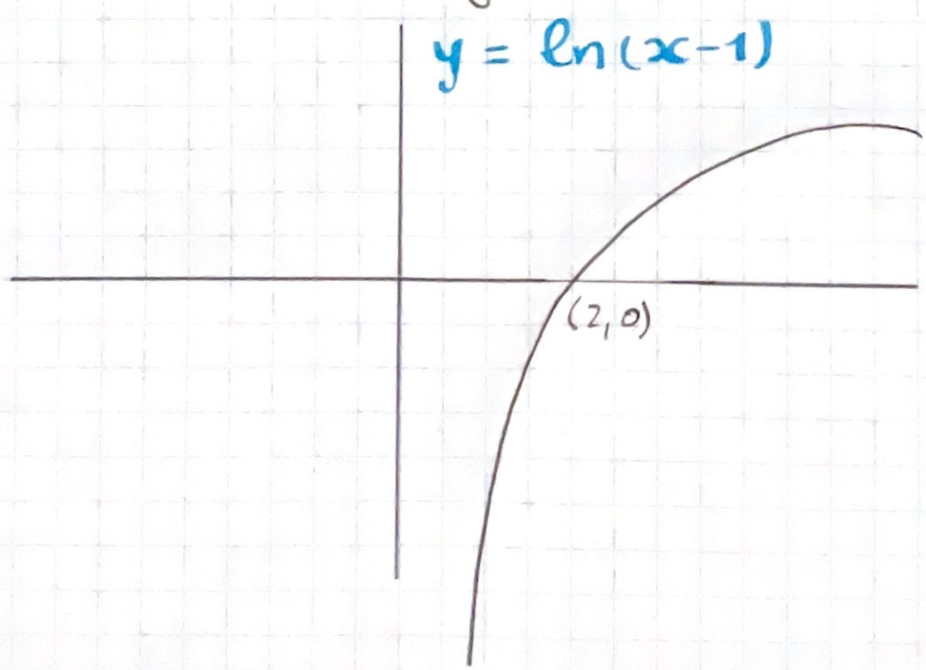
(using your unit circle)

6) (a) $y = \ln(x-1)$

Recall = $\ln(x)$ drawing is

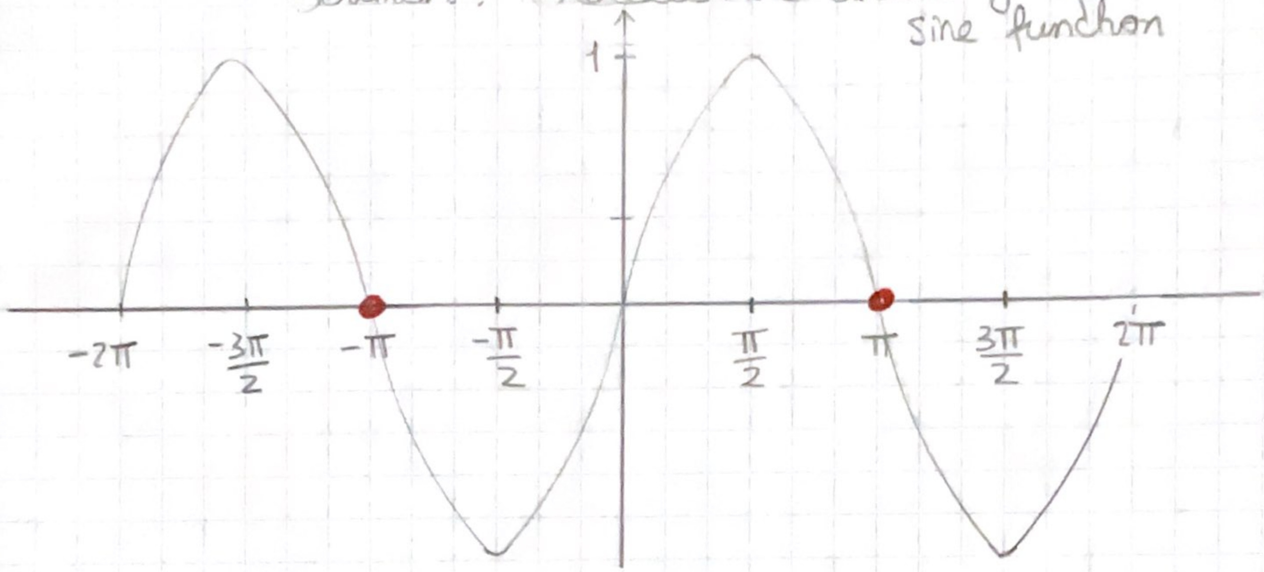


hence $\ln(x-1)$ you shift your graph on the x-axis to the right by 1, hence we get

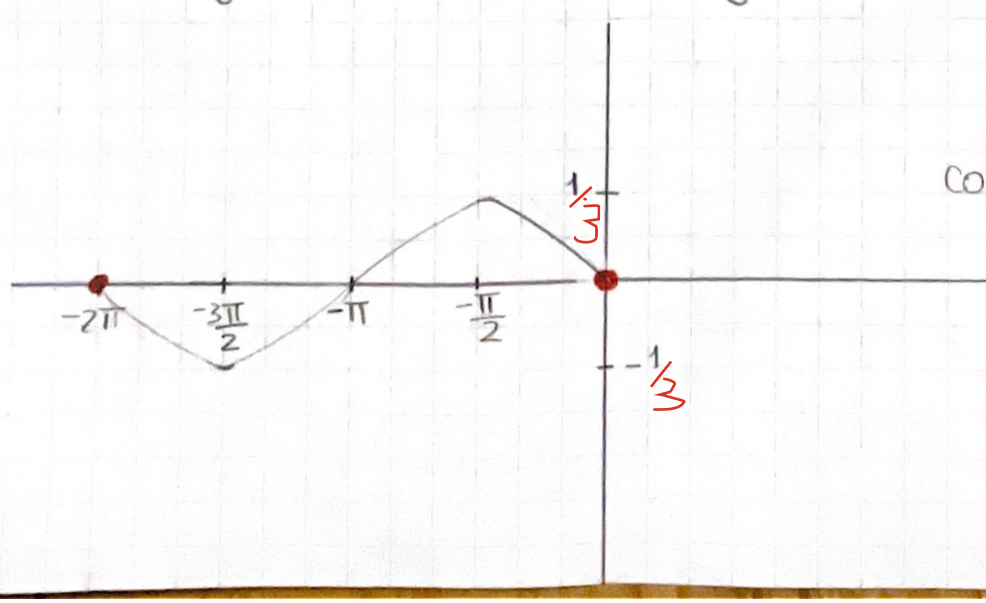


(b) $y = -\frac{1}{3}\sin(x+\pi)$ over the interval $[-\pi, \pi]$.

Solution: Recall the drawing of the sine function



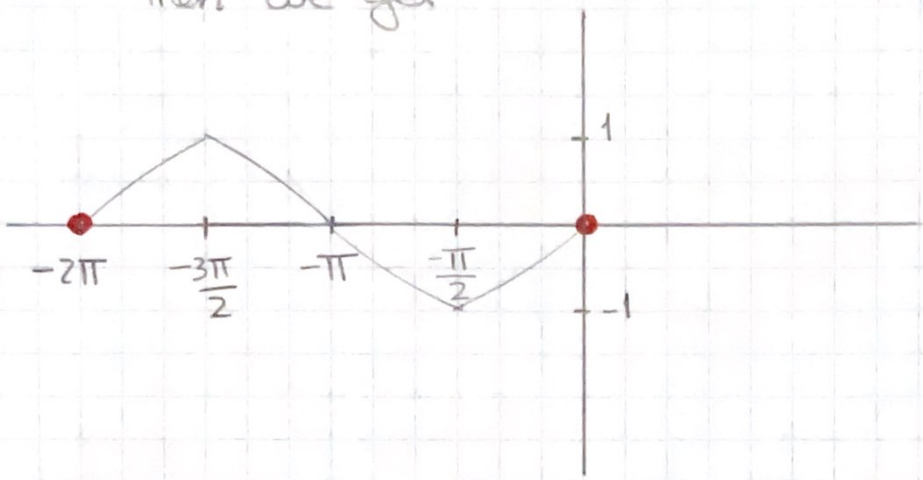
Note that we will work only in the interval $[-\pi, \pi]$, which I put dots on its endpoints above. Now, since we have $(x+\pi)$ as an argument, we will shift our graph to the left by π and then we will shrink our graph by $1/3$, so we will get



complete the drawing on this side by yourself (note you have to stop at π)

Now, it remains to flip the graph upside down as we have -ve sign beside the 1/3.

Then we get

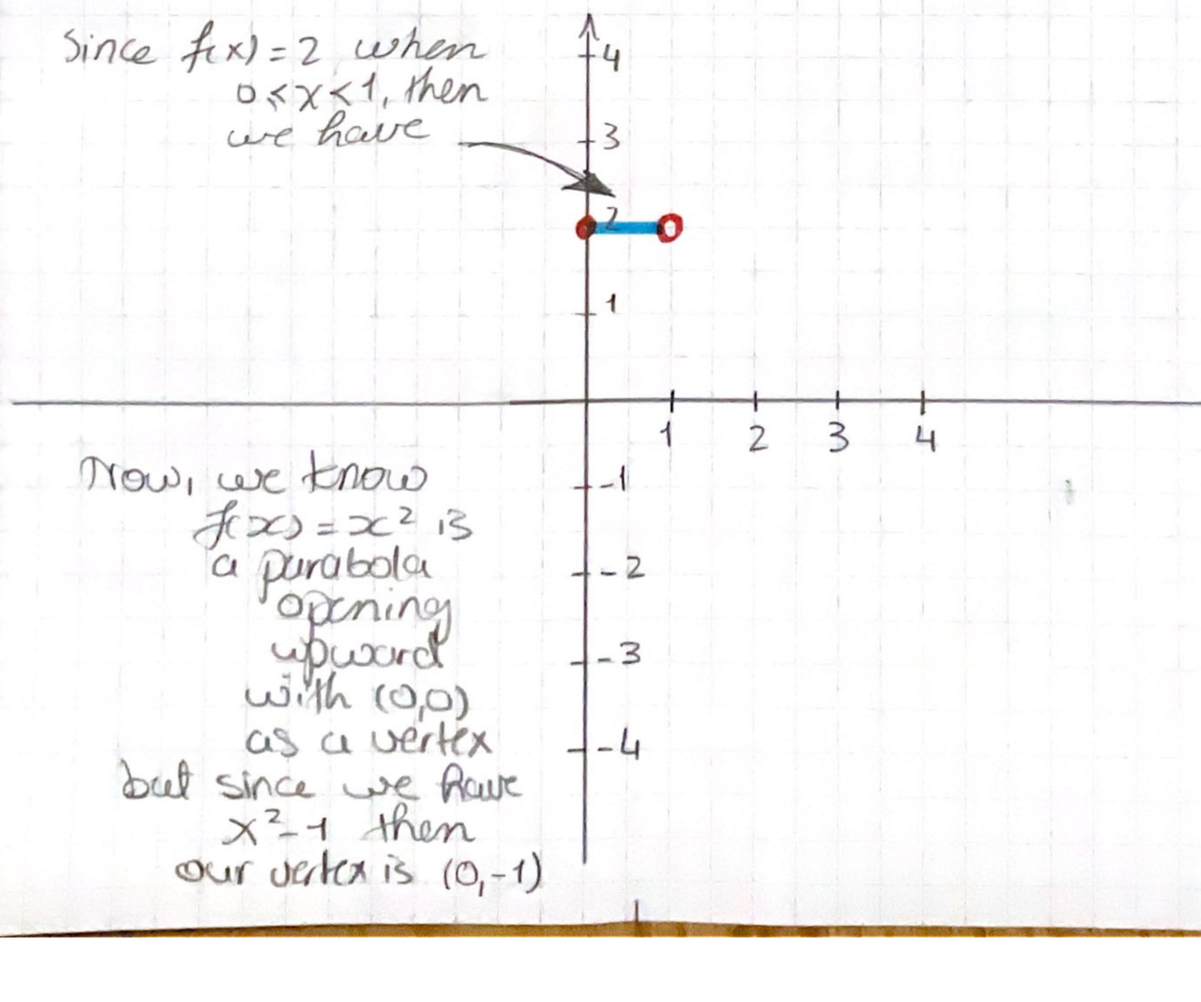


Complete the drawing by yourself on this side (you have to stop at π)

7)

$$f(x) = \begin{cases} x^2-1, & \text{if } x<0 \\ 2, & \text{if } 0 \leq x < 1 \\ \sqrt{x+3}, & \text{if } x>1 \end{cases}$$

Since $f(x)=2$ when $0 \leq x < 1$, then we have



Now, we know $f(x)=x^2$ is a parabola opening upward with (0,0) as a vertex but since we have x^2-1 then our vertex is (0,-1) we also know that the x-intercept of $f(x)=x^2-1$ is as follows:

Put $y=0$ in $y=f(x)=x^2-1$

we get $x^2=1$

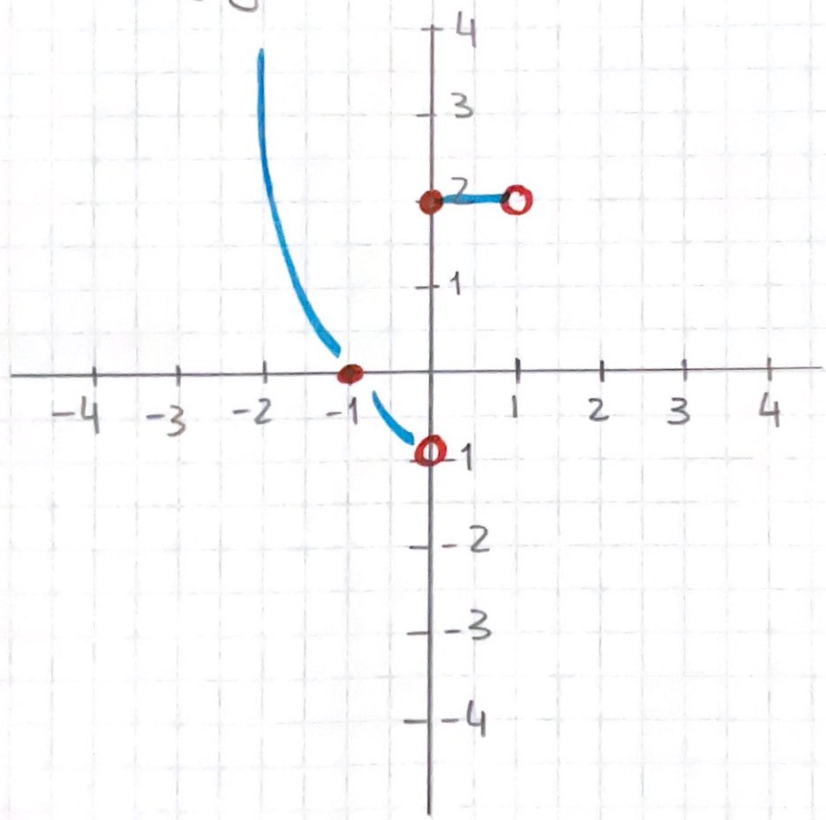
then $x=\pm 1$

but our domain is $x<0$

then $x=-1$ only

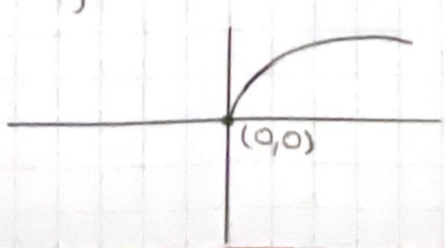
& $(-1,0)$ is the x-intercept

And the figure becomes



Now, it remains to draw $f(x)=\sqrt{x+3}$ if $x>1$

Recall $f(x)=\sqrt{x}$ is drawn as follows



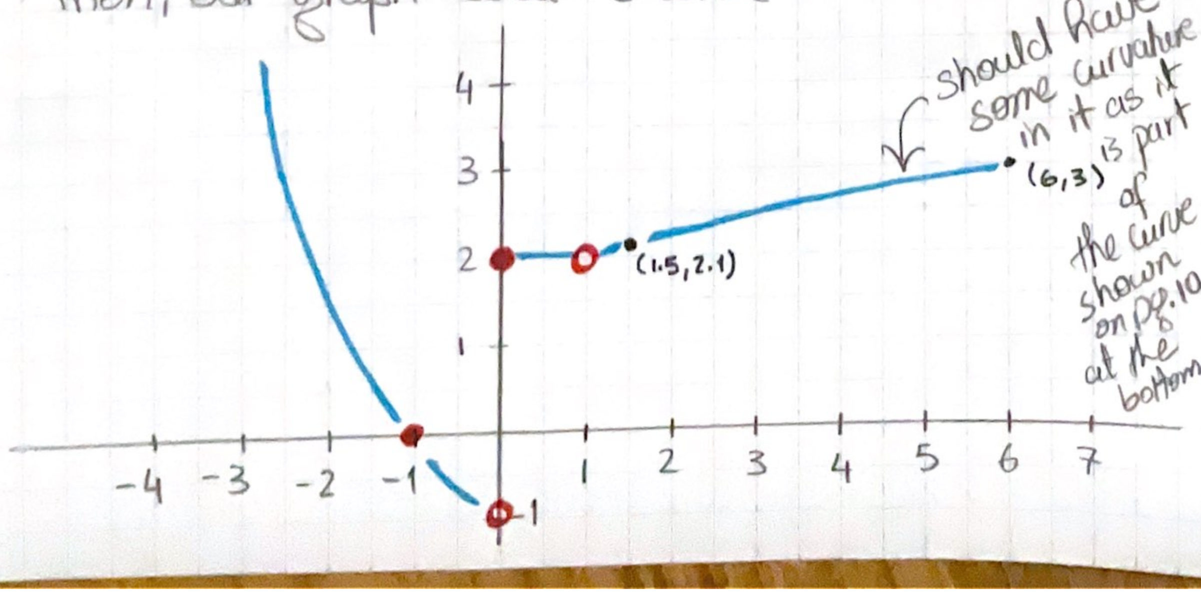
Pg. 32 in calculus book

but here we have $\sqrt{x+3}$ instead, so it is shifted on the x-axis to the left 3 points, but our domain is $x > 1$, so we cannot start from -3, so we will compute points on our function instead starting at $x = 1.1$ (I will use calculator, I don't know in the actual exam what did the students do, maybe they just chosen nice numbers).

also note at $x=1$ $f(x) = 2$ but the circle will be open as 1 is not in my domain.

x	$f(x)$
1.1	2.025
1.5	2.121
6	3

Then, our graph will become



(a) $\lim_{x\to -3} f(x) = \lim_{x\to -3} x^2-1 = 9-1=8$

(b) $\lim_{x\to 0} f(x)$.

Since the function has 2 definitions before and after zero, then have to calculate the limit from the left and from the right.

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} 2 = 2$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} x^2-1 = -1$$

Now, since $\lim_{x\to 0^+} f(x) \neq \lim_{x\to 0^-} f(x)$

then $\lim_{x\to 0} f(x)$ DNE

(c) $f\left(\frac{2}{3}\right)$

since $0 \leq \frac{2}{3} < 1$

then $f(2/3) = 2$.

(d) $\lim_{x \to 1^-} f(x)$

since 1^- lies in $0 \le x < 1$
then $f(x) = 2$

$\therefore \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} 2 = 2$

(e) $\lim_{x \to 1^+} f(x)$

since 1^+ lies in $x > 1$
then $f(x) = \sqrt{x+3}$ and
hence

$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \sqrt{x+3} = \sqrt{4} = 2$

(f) $f(1) =$ undefined as the function is not defined at 1 by the definition given to us.

8)

a) $\lim_{x \to 4} \frac{16-x^2}{x-4}$

Since $16-x^2$ is a difference betn. 2 squares, then we can factorize it into $(4-x)(4+x)$, then

$$\lim_{x \to 4} \frac{16-x^2}{x-4} = \lim_{x \to 4} \frac{(4-x)(4+x)}{(x-4)}$$

$$= \lim_{x \to 4} \frac{-\cancel{(x-4)}(4+x)}{\cancel{(x-4)}}$$

$$= \lim_{x \to 4} -(4+x) = -8$$

b) $\lim_{x \to 1^-} \frac{|x-1|}{x^2-1}$

Since x tends to 1 from the left and since

$$|x-1| = \begin{cases} x-1 & \text{if } x \geq 1 \\ 1-x & \text{if } x < 1 \end{cases},$$

then $|x-1| = 1-x$ in our case here

means therefore

$$\therefore \lim_{x\to 1^-} \frac{|x-1|}{\underbrace{x^2-1}} = \lim_{x\to 1^-} \frac{1-x}{(x-1)(x+1)}$$

difference betn. 2 squares its factorization is $(x-1)(x+1)$

$$= \lim_{x\to 1^-} \frac{-\cancel{(x-1)}}{\cancel{(x-1)}(x+1)}$$

$$= \lim_{x\to 1^-} \frac{-1}{x+1}$$

$$= \frac{-1}{2}$$

(C) $\lim_{h\to 0} \frac{\sqrt{25+h}-5}{h}$

multiplying by the conjugate of the numerator which is $\sqrt{25+h}+5$, then

$$\lim_{h\to 0} \frac{\sqrt{25+h}-5}{h} = \lim_{h\to 0} \frac{(\sqrt{25+h}-5)(\sqrt{25+h}+5)}{h(\sqrt{25+h}+5)}$$

$$= \lim_{h\to 0} \frac{\cancel{25}+\cancel{h}-\cancel{25}}{\cancel{h}(\sqrt{25+h}+5)}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{25+h}+5} = \frac{1}{10}.$$

9) No continuity on the final

Good Luck!