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Stembridge's Phenomenon of Generating Polynomials at q = -1: An Explanation by Concentrating the Homology

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A few years ago, Dennis Stanton asked for a topological explanation for the fact that the Gaussian polynomial, which is the generating function of integer partitions in a rectangle, when evaluated at -1 yields the number of self-complementary partitions. Given a set of combinatorial objects, graded by cardinality or some other ranking, there is a polynomial p(q) which is the generating function of the number of objects of each rank. In some situations there is an involution on the set of objects, such as an involutory symmetry, such that p(-1) equals the number of fixed points of the involution. This phenomenon is known as “Stembridge's q = -1 phenomenon”. Stanton asked for a topological explanation.

Recently John Shareshian, Dennis Stanton, and I provided such an explanation in the case of the Gaussian polynomial by introducing chain complexes whose face numbers are the coefficients in the Gaussian polynomial and whose homology is concentrated in even dimensions, which means that the Euler characteristic can count objects without any cancellation due to its alternating signs. These chain complexes make sense more generally than with rectangular integer partitions, but they do not always have such homology concentration.

As time permits, I will discuss a short, topological proof that these complexes are acyclic in odd dimensions and also a Morse matching lemma that is the main ingredient in the aforementioned homology concentration result. I will also briefly discuss a related chain complex whose face numbers count partitions in a 3-dimensional box and whose homology is again concentrated in dimensions all of the same parity. This complex has a homology basis indexed by semistandard domino tableaux of rectangular shape.
