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# Chapter 14 - Section 8 - Lagrange Multipliers

#### **Section Overview**

In this section we learn how to use the Lagrange Technique to locate extreme (maximum or minimum) values of a multivariable function subject to some constraints. As we shall see, the technique outlined is extremely similar to the method we used in Calculus I to locate extreme values of a single variable function on an interval. We will describe the technique three times, once for a three-variable function with one constraint, once for a three-variable function with two constraints, and finally for the case of an \$n\$-variable function with \$m\$ constraints. We will assume throughout that our objective function is differentiable and our constraints have non-zero gradient, except for a couple examples in which we will discuss how and why the technique may fail when a constraint gradient is the zero vector.

#### Three variable function, single constraint

Let \$ w = f(x,y,z) \$ denote our objective function and \$ g(x,y,z) = k \$ our constraint; that is, we seek for the extreme values attained by the function \$ f \$ on the level surface \$ g = k \$. Suppose \$ f \$ does have an extreme value at a point \$ P = (x\_0,y\_0,z\_0) \$ on the surface \$ g = k \$. Let \$ C \$ be a curve with differentiable parametrization \$ \mathbf{r}(t) \$ that lies on the surface \$ g = k \$ and passes through the point \$ P \$. Let \$ t\_0 \$ denote the parameter value corresponding to the point \$ P \$, so \$ \mathbf{r}(t\_0) = \langle x\_0, y\_0, z\_0 \rangle \$, and let \$ h(t) = f \circ \mathbf{r}(t) \$. Now \$ h \$ has an extreme value at \$ t\_0 \$, so it has a critical point there. Since \$ f \$ is differentiable at \$ P \$ and \$ \mathbf{r} \$ is differentiable, it follows that the composition \$ h \$ is differentiable at \$ t\_0 \$, and so the critical point \$ t\_0 \$ is a root of the derivative. Hence, by the chain rule we have

 $\label{eq:linear} $$ \eqref{linear} $$ \eqref{$ 

Thus the gradient of \$ f \$ at the point \$ P \$ is orthogonal to the line tangent to the curve \$ C \$ at the point \$ P \$. Since this holds for all curves in the surface \$ g = k \$ through the point \$ P \$, it follows that the gradient of \$ f \$ at \$ P \$ is parallel to the direction of the plane tangent to the surface \$ g = k \$ at the point \$ P \$; in other words, at the point \$ P \$, if \$ \nabla g \neq \mathbf{0} \$ then \$ \nabla f = \lambda \nabla g \$ for some number \$ \lambda \$.

Let us recall the technique used in Calculus I to locate extreme values of a differentiable function y = f(x) on an open interval I. If f has an extreme value at a point  $x_0 \in I$ , then  $x_0 \in I$  is a critical point of f, and since we are assuming that f is differentiable on I we must have  $f^{(')}(x_0) = 0$ . We use this result to develop our technique as follows. Assuming that f has extreme values on I, we locate them in two steps:

1. Solve \$ f^{'} = 0 \$

2. Evaluate \$ f \$ at each solution \$ c \$ found in step 1.; largest value is the maximum, smallest is the minimum.

The Lagrange technique works in a very similar way; assuming extreme values exist, we first find critical points, then evaluate at each to determine the extreme values. There is a slight difference in our Lagrange technique though, in that it is not the critical points of f f which we seek for but rather one which involves f f together with our constraint g = k. Let  $\ell = 0$  lambda denote a fourth variable and set  $F = f - \ell (g - k)$ . Critical points of F f are found by solving the vector equation  $\ell = \ell f$ .

four equations with four unknowns:

 $\begin{array}{II} 0 = F_x = f_x - \array 0 = F_y = f_y - \array 0 = F_z = f_z - \array 0 = F_\array 0 = F_$ 

- 1. Solve  $\ F = 0$ .
- 2. Evaluate \$ f \$ at the projections in \$ \mathbb{R}^3 \$ of each solution found in step 1.; largest is maximum, smallest is minimum

 $\ (x,y,z,\lambda) \ (x,y,z,0) \ (x,y,z) \$ 

In words, we replace our fourth coordinate with zero, then identify the point (x,y,z,0) in  $\mathbb{R}^{4}$  with the point (x,y,z) in  $\mathbb{R}^{3}$ .

Example Set One

One

Exercise: Use the Lagrange technique to find the maximum and minimum values of the function  $f(x,y,z) = e^{xyz}$  on the ellipsoid  $2x^2 + y^2 + z^2 = 24$ .

Solution: Set  $F = e^{xyz} - \ln bda (2x^2 + y^2 + z^2 - 24)$ . Then  $\pi + 16F^{0} = \pi F$  yields

 $\begin{array}{II} 0 = yze^{xyz} - 4x \lambda \ 0 = xze^{xyz} - 2y \lambda \ 0 = xye^{xyz} - 2z \lambda \ 0 = xx^2 + y^2 + z^2 - 24 \end{array} \$ 

Multiplying equation one by \$ x \$, equation two by \$ y \$, and equation three by \$ z \$, we see that \$\$ xyze^{xyz} =  $4x^2 \leq 2x^2 \leq 2x^2$ 

Two

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Exercise: Find the maximum and minimum values attained by the function  $f(x,y,z) = x^4 + y^4 + z^4$  on the unit sphere.

Solution: Since the unit sphere is the surface  $x^2 + y^2 + z^2 = 1$  our auxiliary function is  $F(x,y,z,\lambda) = x^4 + y^4 + z^4 - \lambda + y^2 + z^2 - 1)$ , so the vector equation  $\lambda + z^4 - \lambda + y^2 + z^2 - 1$ , so the vector equation  $\lambda + z^4 - \lambda + y^2 + z^2 - 1$ , so the vector equation  $\lambda + z^4 - \lambda + y^4 + z^4 - \lambda + y^2 + z^2 - 1$ .

 $s \eq x^3 - 2x \ = 2x(2x^2 - \ambda) \ 0 = 4y^3 - 2y \ = 2y(2y^2 - \ambda) \ 0 = 4z^3 - 2z \ = 2y(2y^2 - \ambda) \ 1 = x^2 + y^2 + z^2 \ = darray \$ 

Using a table to organize the possible cases we quickly locate the maximum value of 1 and the minimum value of  $\frac{1}{3}$ :

### Three variable function, two constraints

Let f(x,y,z) be our objective function and g(x,y,z) = k and h(x,y,z) = 0 but two constraints. Let  $F = f - \lambda (g - k) - \lambda (h - 0)$  on the intersection of the level surfaces g = k and h = 0 but  $h + \lambda (x_0, y_0, z_0)$  on the intersection of the level surfaces g = k and h = 0 but  $h + \lambda (x_0, y_0, z_0)$  on the intersection of the  $\lambda (x_0, y_0, z_0)$  on the intersection of the surfaces g = k and h = 0 but  $h + \lambda (x_0, y_0, z_0)$  on the intersection of the level surfaces g = k and h = 0 but  $h + \lambda (x_0, y_0, z_0)$  on the intersection of  $g = \lambda (x_0, y_0, z_0)$  but h = 0 but h =

Example Set Two

One

Exercise: Find the extreme values of  $z \pm z = 24 \pm z^2 \pm z^$ 

Solution: Set  $F = z - \lambda (x^2 + y^2 - z^2) - \lambda (x + y + z - 24)$  and solve the vector equation  $\lambda (x^2 + y^2 - z^2) - \lambda (x + y + z - 24)$  and solve the vector equation  $\lambda (x^2 + y^2 - z^2) - \lambda (x + y + z - 24)$ 

We see if \$ x = 0 \$ then \$ \mu = 0 \$, so \$ y = 0 \$ or \$ \lambda = 0 \$. If \$ y = 0 \$ then \$ z = 0 \$ by equation four, so equation three is \$ 0 = 1 \$ a contradiction. If \$ \lambda = 0 \$ then again we have \$ 0 = 1 \$ for equation three, a contradiction. Hence, \$ x \neq 0 \$. Similarly, we find \$ y,z, \lambda , \mu \neq 0 \$. Now the first three equations give \$ x = y =  $\frac{1}{2} \left[1 - \frac{1}{2} \right] \left[1 - \frac{1}{2} \left[1 - \frac{1}{2} \right] \left[1 - \frac$ 

## \$n\$ variable function, \$m\$ constraints

Example Set Three

One

Exercise: Find the maximum and minimum values attained by summing the coordinates of a point on the unit \$n\$-sphere.

Solution: Our objective function is  $f(x_1,x_2,|dots,x_n) = x_1 + x_2 + |dots + x_n \ and our constraint is <math>x_1^2 + x_2^2 + |dots + x_n^2 = 1 \ so we have \ F = x_1 + x_2 + |dots + x_n - |ambda (x_1^2 + x_2^2 + |dots + x_n^2 - 1) \ we see that for each \ 1 | leq i | leq n \, we have \ frac{\partial F}{\partial x_i} = 1 - 2 | lambda x_i \, and so the vector equation \ mathbf{0} = |nabla F \ yields \ x_i = |frac{1}{2 | lambda} \ for all \ i \ we have \ frac{1}{2 | lambda} \ for all \ i \ begin{tabular}{lambda}{la$ 

→ Chapter 15: Multiple Integrals

Discussion

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