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Chapter 14 - Section 8 - Lagrange Multipliers

Section Overview

In this section we learn how to use the Lagrange Technique to locate extreme (maximum or minimum) values of a multivariable function subject to some constraints. As we shall see, the technique outlined is extremely similar to the method we used in Calculus I to locate extreme values of a single variable function on an interval. We will describe the technique three times, once for a three-variable function with one constraint, once for a three-variable function with two constraints, and finally for the case of an \$n\$-variable function with \$m\$ constraints. We will assume throughout that our objective function is differentiable and our constraints have non-zero gradient, except for a couple examples in which we will discuss how and why the technique may fail when a constraint gradient is the zero vector.

Three variable function, single constraint

Let \$ w = f(x,y,z) \$ denote our objective function and \$ g(x,y,z) = k \$ our constraint; that is, we seek for the extreme values attained by the function \$ f \$ on the level surface \$ g = k \$. Suppose \$ f \$ does have an extreme value at a point $\$ P = (x_0,y_0,z_0) \$$ on the surface \$ g = k \$. Let \$ C \$ be a curve with differentiable parametrization $\$ \mathbb{P} (t) \$$ that lies on the surface \$ g = k \$ and passes through the point \$ P \$. Let $\$ t_0 \$$ denote the parameter value corresponding to the point \$ P \$, so $\$ \mathbb{P} (t_0) = \mathbb{P} (t_0) = \mathbb{P} (t_0) *$, and let $\$ h(t) = f \mathbb{P} (t_0) *$. Now \$ h \$ has an extreme value at $\$ t_0 \$$, so it has a critical point there. Since \$ f \$ is differentiable at \$ P \$ and $\$ \mathbb{P} (t_0) \$$ is differentiable, it follows that the composition \$ h \$ is differentiable at $\$ t_0 \$$, and so the critical point $\$ t_0 \$$ is a root of the derivative. Hence, by the chain rule we have

 $$\ \end{aligned} 0 \&= h^{'}(t_0) \& = \frac{\pi c^{\beta t} (x_0,y_0,z_0) \frac{dx}{dt} (t_0) + \frac{dy}{dt} (t_0) + \frac{dy}{dt} (t_0) + \frac{dy}{dt} (t_0) + \frac{dy}{dt} (t_0) \\ \end{aligned} $$$

Thus the gradient of \$ f \$ at the point \$ P \$ is orthogonal to the line tangent to the curve \$ C \$ at the point \$ P \$. Since this holds for all curves in the surface \$ g = k \$ through the point \$ P \$, it follows that the gradient of \$ f \$ at \$ P \$ is parallel to the direction of the plane tangent to the surface \$ g = k \$ at the point \$ P \$; in other words, at the point \$ P \$, if \$ \nabla g \neq \mathbf{0} \$ then \$ \nabla f = \lambda \nabla g \$ for some number \$ \lambda \$.

Let us recall the technique used in Calculus I to locate extreme values of a differentiable function \$y = f(x) \$ on an open interval \$I \$. If \$f \$ has an extreme value at a point $$x_0 \in I $$, then $$x_0 \in I $$ is a critical point of \$f \$, and since we are assuming that \$f \$ is differentiable on \$I \$ we must have $$f^{'}(x_0) = 0 $$. We use this result to develop our technique as follows. Assuming that \$f \$ has extreme values on \$I \$, we locate them in two steps:

- 1. Solve $f^{'} = 0$
- 2. Evaluate \$ f \$ at each solution \$ c \$ found in step 1.; largest value is the maximum, smallest is the minimum.

The Lagrange technique works in a very similar way; assuming extreme values exist, we first find critical points, then evaluate at each to determine the extreme values. There is a slight difference in our Lagrange technique though, in that it is not the critical points of f which we seek for but rather one which involves f together with our constraint g = k. Let αf lambda denote a fourth variable and set f = f - \lambda (g - k) f. Critical points of f are found by solving the vector equation f \text{ mathbf} and f \text{ which yields the following system of }

four equations with four unknowns:

 $\label{eq:continuous} $$ \left\{II\right\} 0 = F_x = f_x - \lambda g_x \setminus 0 = F_y = f_y - \lambda g_y \setminus 0 = F_z = f_z - \lambda g_x \setminus 0 = F_\lambda g_x + g_y = g_y + g_y + g_y = g_y + g_y + g_y = g_y + g_y + g_y + g_y = g_y + g_y +$

- 1. Solve $\ \nabla \mathbf{F} = 0 \$.
- 2. Evaluate \$ f \$ at the projections in \$ \mathbb{R}^3 \$ of each solution found in step 1.; largest is maximum, smallest is minimum

Note that in step 1. above, we don't actually need the $\$ \lambda \\$ part of the solution. You may find it's values in the course of finding \\$ x,y, \\$ and \\$ z \\$, but more often it will just be used to help relate the important variables. The points found in step 1. are in four-space, but we need points in three space to evaluate \\$ f \\$; specifically, we need the \\$ x,y,z \\$ part. The projection referred to above works as follows:

 $\$ (x,y,z,\lambda) \rightarrow (x,y,z,0) \simeg (x,y,z) \$\$

In words, we replace our fourth coordinate with zero, then identify the point (x,y,z,0) in \mathbb{R}^4 with the point (x,y,z) in \mathbb{R}^3 .

Example Set One

One

Exercise: Use the Lagrange technique to find the maximum and minimum values of the function $f(x,y,z) = e^{xyz}$ on the ellipsoid $2x^2 + y^2 + z^2 = 24$.

Solution: Set $F = e^{xyz} - \lambda (2x^2 + y^2 + z^2 - 24)$. Then $\theta = nabla F$ yields

 $\$ \begin{array}{II} 0 = yze^{xyz} - 4x \lambda \ 0 = xze^{xyz} - 2y \lambda \ 0 = xye^{xyz} - 2z \lambda \ 2x^2 + y^2 + z^2 - 24 \lambda \

Multiplying equation one by \$ x \$, equation two by \$ y \$, and equation three by \$ z \$, we see that \$\$ xyze^{xyz} = $4x^2 \le 2^2 \le 2^$

Two

Exercise: Find the maximum and minimum values attained by the function $f(x,y,z) = x^4 + y^4 + z^4$ on the unit sphere.

Solution: Since the unit sphere is the surface $x^2 + y^2 + z^2 = 1$ our auxiliary function is $F(x,y,z,\lambda) = x^4 + y^4 + z^4 - \lambda (x^2 + y^2 + z^2 - 1)$, so the vector equation $\lambda (x^2 + y^2 + z^2 - 1)$ so the vector equation $\lambda (x^2$

 $\frac{1}{0} = 4x^3 - 2x \lambda = 2x(2x^2 - \lambda) \ 0 = 4y^3 - 2y \lambda = 2y(2y^2 - \lambda) \ 0 = 4z^3 - 2z \lambda = 2z(2z^2 - \lambda) \ 1 = x^2 + y^2 + z^2 \lambda$

Using a table to organize the possible cases we quickly locate the maximum value of 1 and the minimum value of $\frac{1}{3}$:

Three variable function, two constraints

Let f(x,y,z) be our objective function and g(x,y,z) = k and $h(x,y,z) = \ell$ our two constraints. Let f = f - \lambda g = k and g

Example Set Two

One

Exercise: Find the extreme values of \$z\$ subject to the constraints $\$x^2 + y^2 = z^2\$$ and \$x + y + z = 24\$.

Solution: Set $F = z - \lambda (x^2 + y^2 - z^2) - \mu (x + y + z - 24)$ and solve the vector equation $\hbar (0) = \hbar F$. We get the system of equations

 $\$ \begin{array}{||} 0 = 2x \lambda + \mu \\ 0 = 2y \lambda + \mu \\ 0 = 1 - 2z \lambda - \mu \\ z^2 = x^2 + y^2 \\ 24 = x + y + z \end{array} \$\$

We see if \$ x = 0 \$ then \$ \mu = 0 \$, so \$ y = 0 \$ or \$ \lambda = 0 \$. If \$ y = 0 \$ then \$ z = 0 \$ by equation four, so equation three is \$ 0 = 1 \$ a contradiction. If \$ \lambda = 0 \$ then again we have \$ 0 = 1 \$ for equation three, a contradiction. Hence, \$ x \neq 0 \$. Similarly, we find \$ y,z, \lambda , \mu \neq 0 \$. Now the first three equations give \$ x = y = \frac{-\mu}{2 \lambda} \$ and \$ z = \frac{1 - \mu}{2 \lambda} \$. Substituting into equations four and five we have \$ \frac{(1 - \mu)^2}{4 \lambda^2} = \frac{\mu^2}{2 \lambda^2} \$ and \$ 24 = \frac{1 - 3 \mu}{2 \lambda} \$, from which we find two solutions \$ \mu = -1 + \sqrt{2} \$ and \$ 2 \lambda = \frac{4 - 3 \sqrt{2}}{24} \$ and \$ \mu = -1 - \sqrt{2} \$ and \$ 2 \lambda = \frac{4 + 3 \sqrt{2}}{24} \$ and so we have our minimum value \$ z = -24(1 + \sqrt{2}) \$ when \$ \mu = -1 - \sqrt{2} \$.

\$n\$ variable function, \$m\$ constraints

As usual we form an auxiliary function $F \$ with of objective and all constraints and seek for solutions to the vector equation $\$ \mathbf{0} = \nabla F \\$. Let \\$ x_1,x_2,\ldots,x_n \\$ denote the \\$n\\$-variables and \\$ g_1 = k_1, g_2 = k_2, \ldots, g_m = k_m \\$ the \\$ m \\$ constraints. Then we introduce \\$m\\$ variables \\$ \lambda_1,\lambda_2,\ldots,\lambda_m \\$ and set \\$ F = f - \displaystyle \sum_{i=1}^m \lambda_i (g_i - k_i) \\$\$

Example Set Three

One

Exercise: Find the maximum and minimum values attained by summing the coordinates of a point on the unit \$n\$-sphere.

Solution: Our objective function is \$ $f(x_1,x_2,\lambda,x_n) = x_1 + x_2 + \lambda + x_n$ \$ and our constraint is \$ $x_1^2 + x_2^2 + \lambda + x_n^2 = 1$ \$, so we have \$ $F = x_1 + x_2 + \lambda + x_n - \lambda + x_n^2 = 1$ \$, so we have \$ $F = x_1 + x_2 + \lambda + x_n - \lambda + x_n^2 = 1$ \$. We see that for each \$ 1 \leq i \leq n \$, we have \$ \frac{\partial F}{\partial x_i} = 1 - 2 \lambda x_i \$, and so the vector equation \$ \mathbb{6} = \mathbb{6} \$ for all \$ i \$. Hence, from our last equation we have \$ 1 = n(\frac{1}{4 \lambda} + \lambda + x_n^2 +

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