Math 504 - Old Homework

$\newcommand{\aut}{\textrm{Aut}} \newcommand{\end}{\textrm{End}} \newcommand{\sub}{\textrm{Sub}} \newcommand{\min}{\textrm{min}} \newcommand{\lub}{\textrm{l.u.b.}} \newcommand{\glb}{\textrm{g.l.b.}} \newcommand{\join}{\vee} \newcommand{\bigjoin}{\bigvee} \newcommand{\meet}{\wedge} \newcommand{\bigmeet}{\bigwedge} \newcommand{\normaleq}{\unlhd} \newcommand{\normal}{\lhd} \newcommand{\union}{\cup} \newcommand{\intersection}{\cap} \newcommand{\bigunion}{\bigcup} \newcommand{\bigintersection}{\bigcap} \newcommand{\sq}[2][\ ]{\sqrt[#1]{#2\,}} \newcommand{\pbr}[1]{\langle #1\rangle} \newcommand{\ds}{\displaystyle} \newcommand{\C}{\mathbb{C}} \newcommand{\R}{\mathbb{R}} \newcommand{\Q}{\mathbb{Q}} \newcommand{\Z}{\mathbb{Z}} \newcommand{\N}{\mathbb{N}} \newcommand{\A}{\mathbb{A}} \newcommand{\F}{\mathbb{F}} \newcommand{\T}{\mathbb{T}} \newcommand{\ol}[1]{\overline{#1}} \newcommand{\ul}[1]{\underline{#1}} \newcommand{\imp}{\Rightarrow} \newcommand{\rimp}{\Leftarrow} \newcommand{\pinfty}{1/p^\infty} \newcommand{\power}{\mathcal{P}} \newcommand{\calL}{\mathcal{L}} \newcommand{\calC}{\mathcal{C}} \newcommand{\calN}{\mathcal{N}} \newcommand{\calB}{\mathcal{B}} \newcommand{\calF}{\mathcal{F}} \newcommand{\calR}{\mathcal{R}} \newcommand{\calS}{\mathcal{S}} \newcommand{\calU}{\mathcal{U}} \newcommand{\calT}{\mathcal{T}} \newcommand{\gal}{\textrm{Gal}} \newcommand{\isom}{\approx} \renewcommand{\hom}{\textrm{Hom}} $

Problem Set 11 Due 04/20/2018 (complete)

1. Let G be a group with identity element e. Let $\calB_e$ be a collection of subgroups of G which form a basis for the neighborhoods of e. Show that the basic open sets induced by $\calB_e$ are clopen sets, i.e. closed and open.
2. Prove that the normal closure of a finite separable extension over a field K is a finite Galois extension.
3. Given a projective system $(G_i|i\in I)$ with $(\rho_{i,j}|i\leq j)$ of groups, show that the following subset of the product \[\left\{a\in\prod_{i\in I}G_i\middle|\rho_{i,j}(a_j)=a_i \text{ for all } i\leq j\right\} \] together with the projection map into each factor, form a projective limit for the given system.
4. Prove the last claim in the proof that closed under the Krull topology is equivalent to closed under the $\alpha\beta$ closure operator. That is, if $N\in\calF_{\calN}$ and $H\leq G$, then $HN\in\calF$.

Problem Set 10 Due 04/13/2018 (complete)

1. Show that the directed union (ordered by divisibility on $\N^+$)\[\bigunion_{n\in\N^+}\F_{p^n} \] is the algebraic closure of $\F_p$.
2. Show that σ_S, as defined in class on 03/30/18, is an automorphism of $F/\Q$, the field extension in McCarthy's example.
3. Let G be a group with identity element e. Let $\calB_e$ be a collection of subgroups of G which form a basis for the neighborhoods of e. Show that the collection \[\{gH|\ g\in G, H\in\calB_e\}, \] of all left cosets of the subgroups in $\calB_e$ is a basis for a topology on G.
4. Let G be a group, and $H_1,H_2\leq G$. Show that \[[G:H_1\intersection H_2] \leq [G:H_1][G:H_2]. \]

Problem Set 9 Due 03/30/2018 (complete)

1. Prove or disprove: the lattice of centralizers of a group G is a sublattice of $\sub(G)$, the lattice of subgroups of G.

2. Write the details in the proof of the lemma stated in class: For sets A, B, a binary relation $\rho\subseteq A\times B$ from A to B induces a Galois connection between $\power(A)$ and $\power(B)$.
3. Let P be a poset with smallest element 0. For $x,x^{*}\in P$ we say that x^* is a pseudo-complement of x if
   - $x\meet x^* =0$, and
   - $x\meet y = 0 \imp y \leq x^*$.
     Show that:
   I. if x has a pseudo-complement in P, then it is unique,
   II. if P is pseudo-complement, i.e. every element of P has a pseudo-complement, then P with the map $\alpha:x\mapsto x^*$ form a symmetric Galois connection.

Problem Set 8 Due 03/23/2018 (complete)

1. Show that if E/K is separable then $\mbox{$[E:K]_s =_f [E:K]$}$, where $=_f$ means both sides are finite and equal, or both are infinite. Note that this and its converse were proved in class for finite extensions. Show that the converse is not true in general.
2. Prove or disprove: all cyclotomic polynomials have all their coefficients in $\{0,\pm 1\}$.
3. Show that if n is even then \[\phi_{2n}(x) = \phi_n(x^2), \] and if $n\geq 3$ is odd then \[\phi_{2n}(x) = \phi_n(-x). \]
4. Let P be a locally finite poset. For $y\neq x\in P$, \[\sum_{y\leq \ul{z}\leq x}\mu(z,x)=0 \] Hint: Fix $y\in P$, and then use induction on the Artinian poset \[\{u\in P\mid u > y\}. \]

Problem Set 7 Due 03/16/2018 (complete)

1. Let E/K be an algebraic extension, and let $E_i=E\intersection K^{\pinfty}$. Prove or disprove that E/E_i is separable. (Hint: try first the case $E=\ol{K}$)
2. Each $\varphi\in\aut_K(\ol{K})$ induces a complete lattice automorphism on $\sub_K(\ol{K})$.
3. Let S be a set, and $P(x,B)$ denote a property, where $x \in S$ and $B ⊆ S$. When $P(x,B)$ is true, we will say that x has the property P, with respect to B. For $A,B ⊆ S$, write $P(A,B)$ provided all elements of A have property P w.r.t. B, i.e. for all $x∈A$, we have $P(x,B)$. Let $$B^P := \{x\in S\ |\ P(x,B)\}$$ be the set of elements of S related to B via the property P. Assume the property P satisfies:
   I. All elements of B satisfy property P w.r.t. B, i.e. $x ∈ B ⇒ P(x,B)$,
   II. if x has property P w.r.t. B, and $B ⊆ A$, then x has property P w.r.t. A, i.e. $(B ⊆ A \textrm{ and } P(x,B))⇒P(x,A)$,
   III. if x has property P w.r.t. A, and $P(A,B)$, then x has property P w.r.t. B, i.e. $P(x,A) \textrm{ and } P(A,B) ⇒ P(x,B)$.
      Show that the map $B \mapsto B^P$ is a closure operator.
4. Let E/K be an algebraic extension. The normal closure of E/K is the spliting field of the set of polynomials \[\displaystyle A = \{{\min}_{K}(\alpha)\mid\alpha\in E^\times\}. \]

Problem Set 6 Due 03/09/2018 (complete)

1. Let K be a field of prime characteristic p. The field K^{1/p^∞} is the smallest perfect field that contains K.
2. Let $K=\F_2(s,t)$ be the field of rational functions in two variables s and t, over the two element field, $\F_2$. Let $\alpha=\sqrt{s}$ and $\beta=\sqrt{t}$, i.e. α is a root of $x^2-s\in K[x]$, and similarly for β. Let $F=K(\alpha,\beta)$. Find $[F:K]$. Show that any $\gamma\in F$ has degree 1 or 2 over K.

Problem Set 5 Due 02/23/2018 (complete)

1. Show that a finite subgroup of the multiplicative group $K^\times$ of any field K is cyclic.
2. If K is a perfect field and F/K is an algebraic extension then F is perfect.
3. Let $K\leq E\leq F$, and $\alpha\in F$, algebraic over K. Prove:
   I. If α is separable over K, then it is separable over E.
   II. If α is separable over E, and E/K is separable, then α is separable over K.
4. Let $K=\F_2(s,t)$ be the field of rational functions in two variables s and t, over the two element field, $\F_2$. Let $\alpha=\sqrt{s}$ and $\beta=\sqrt{t}$, i.e. α is a root of $x^2-s\in K[x]$, and similarly for β. Prove or disprove that $K(\alpha,\beta)$ is a simple extension of K.

Problem Set 4 Due 02/16/2018 (complete)

1. Show that the algebraic closure is a *closure operator*, i.e.
   I. $K\leq\ol{K}$,

   II. $\ol{\ol{K}} =\ol{K}$,

   III. $K\leq E \imp \ol{K}\leq\ol{E}$.

2. Let F/K be a field extension, and $\varphi:F\to L$ a field homomorphism. Let $\widehat{F}=\varphi(F)$ and $\widehat{K}=\varphi(K)$. Prove:
   I. $[\widehat{F}:\widehat{K}]=[F:K]$.
   II. If F/K is algebraic, then so is $\widehat{F}/\widehat{K}$.
   III. If F/K is transcendental, then so is $\widehat{F}/\widehat{K}$.
   IV. If F is an algebraic closure of K, then $\widehat{F}$ is an algebraic closure of $\widehat{K}$.
3. Let $\ol{K}$ be an algebraic closure of K. Show that $\ol{K}$ is minimal with the property of being an extension of K which is algebraically closed.
4. Let F/K be a field extension and $E,L\in\sub_K(F)$. Show that if E/K is algebraic then EL is algebraic over L. If EL is algebraic over L, does it follow that E is algebraic over K? How about $E/(E\intersection L)$?

Problem Set 3 Due 02/09/2018 (complete)

1. Prove the corollary stated in class: If $K\leq E_i\leq F$ and each E_i/K is algebraic, then the join $\displaystyle\bigjoin_{i\in I}E_i$ is algebraic over K.
2. Let F/K be a finite extension. Prove that every endomorphism of F that fixes K is an automorphism of F.
3. Consider the extension $F=\Q(\alpha,\omega)$ of $\Q$ discussed in class, where α is a root of x^3-2 and ω is a root of x^2+x+1. Construct several automorphisms of F. Is there a bound for the number of automorphisms of F?

Problem Set 2 Due 02/02/2018 (complete)

1. Show that the direct (cartesian) product of two fields is never a field.
2. Show that $\Q(\sq{2})\neq\Q(\sq{3})$. Generalize.
3. Page 163, IV.2.1
4. Page 163, IV.2.2, IV.2.4

Problem Set 1 Due 01/26/2018 (complete)

1. Let G be a group and $N\normaleq G$. G is solvable iff N and G/N are solvable. In this case, $$l(G) \leq l(N) + l(G/N)$$
2. If L is a poset in which every subset has a l.u.b., then every subset of L also has a g.l.b.
3. Given a lattice $(L,\meet,\join)$ in the algebraic sense, show that the binary relation defined by $$ x \leq y \quad iff \quad x \meet y = x $$ is a partial order on L, and for $x,y \in L$, $x \meet y$ is the g.l.b.{x,y}, and $x \join y$ is the l.u.b.{x,y}.
4. Let A be a universal algebra, and $\sub(A)$ the complete lattice of subuniverses of A. If $D\subseteq\sub(A)$ is directed, then $\ds\left(\bigunion_{X\in D}X\right)\in\sub(A)$.